

ON LESIEUR'S MEASURED QUANTUM GROUPOIDS

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ABSTRACT. In his thesis ([L1]), which is published in an expended and revised version ([L2]), Franck Lesieur had introduced a notion of measured quantum groupoid, in the setting of von Neumann algebras, using intensively the notion of pseudo-multiplicative unitary, which had been introduced in a previous article of the author, in collaboration with Jean-Michel Vallin [EV]. In [L2], the axioms given are very complicated and are here simplified.

1. INTRODUCTION

1.1. In two articles ([Val1], [Val2]), J.-M. Vallin has introduced two notions (pseudo-multiplicative unitary, Hopf-bimodule), in order to generalize, up to the groupoid case, the classical notions of multiplicative unitary [BS] and of Hopf-von Neumann algebras [ES] which were introduced to describe and explain duality of groups, and led to appropriate notions of quantum groups ([ES], [W1], [W2], [BS], [MN], [W3], [KV1], [KV2], [MNW]).

In another article [EV], J.-M. Vallin and the author have constructed, from a depth 2 inclusion of von Neumann algebras $M_0 \subset M_1$, with an operator-valued weight T_1 verifying a regularity condition, a pseudo-multiplicative unitary, which led to two structures of Hopf bimodules, dual to each other. Moreover, we have then constructed an action of one of these structures on the algebra M_1 such that M_0 is the fixed point subalgebra, the algebra M_2 given by the basic construction being then isomorphic to the crossed-product. We construct on M_2 an action of the other structure, which can be considered as the dual action.

If the inclusion $M_0 \subset M_1$ is irreducible, we recovered quantum groups, as proved and studied in former papers ([EN], [E1]).

Therefore, this construction leads to a notion of "quantum groupoid", and a construction of a duality within "quantum groupoids".

1.2. In a finite-dimensional setting, this construction can be mostly simplified, and is studied in [NV1], [BSz1], [BSz2], [Sz], [Val3], [Val4], and examples are described. In [NV2], the link between these "finite quantum groupoids" and depth 2 inclusions of II_1 factors is given.

1.3. Franck Lesieur introduced in his thesis [L1] a notion of "measured quantum groupoids", in which a modular hypothesis on the basis is required. Mimicking in a wider setting the technics of Kustermans and Vaes [KV], he obtained then a pseudo-multiplicative unitary, which, as in the quantum group case, "contains" all the information of the object (the von Neuman algebra, the coproduct, the antipod, the co-inverse). Unfortunately, the axioms chosen then by Lesieur don't fit perfectly with the duality (namely, the dual object doesnot fit the modular condition on the basis chosen in [L1]), and, for this purpose, Lesieur gave the name of "measured quantum groupoids" to a wider class [L2], whose axioms could be described as the analog of [MNW], in which a duality is defined and studied, the initial objects considered in [L1] being denoted now "adapted measured quantum groupoids". In [E3] had been shown that, with suitable conditions, the objects constructed in [EV] from depth 2 inclusions, are "measured quantum groupoids" in this new setting.

1.4. Unfortunately, the axioms given in ([L2], 4) are very complicated, and there was a serious need for simplification. This is the goal of this article.

1.5. This article is organized as follows :

In chapter 2 are recalled all the definitions and constructions needed for that theory, namely Connes-Sauvageot's relative tensor product of Hilbert spaces, fiber product of von Neumann algebras, and Vaes' Radon-Nikodym theorem.

The chapter 3 is a résumé of Lesieur's basic result ([L2], 3), namely the construction of a pseudo-multiplicative unitary associated to a Hopf-bimodule, when exist a left-invariant operator-valued weight, and a right-invariant valued weight.

The chapter 4 is mostly inspired from Lesieur's "adapted measured quantum groupoids" ([L2], 9), with a wider hypothesis, namely, that there exists a weight on the basis such that the modular automorphism groups of two lifted weights (via the two operator-valued weights) commute. This hypothesis allows us to use Vaes' theorem, and is a nice generalization of the existence of a relatively invariant measure on the basis of a groupoid. With that hypothesis, mimicking ([L2], 9), we construct a co-inverse and a scaling group.

In chapter 5, we go on with the same hypothesis. It allows us to construct two automorphism groups on the basis, which appear to be invariant under the relatively invariant weight introduced in chapter 4. It is then straightforward to get that we are now in présence of Lesieur's "measured quantum groupoids" ([L2], 4) and chapter 6 is devoted to main properties of these.

1.6. The author is indebted to Frank Lesieur, Stefaan Vaes, Leonid Vainerman, and especially Jean-Michel Vallin, for many fruitful conversations.

2. PRELIMINARIES

In this chapter are mainly recalled definitions and notations about Connes' spatial theory (2.1, 2.3) and the fiber product construction (2.4, 2.5) which are the main technical tools of the theory of measured quantum theory.

2.1. Spatial theory [C1], [S2], [T]. Let N be a von Neumann algebra, and let ψ be a faithful semi-finite normal weight on N ; let $\mathfrak{N}_\psi, \mathfrak{M}_\psi, H_\psi, \pi_\psi, \Lambda_\psi, J_\psi, \Delta_\psi, \dots$ be the canonical objects of the Tomita-Takesaki construction associated to the weight ψ . Let α be a non-degenerate normal representation of N on a Hilbert space \mathcal{H} . We may as well consider \mathcal{H} as a left N -module, and write it then ${}_N\mathcal{H}$. Following ([C1], definition 1), we define the set of ψ -bounded elements of ${}_N\mathcal{H}$ as :

$$D({}_N\mathcal{H}, \psi) = \{\xi \in \mathcal{H}; \exists C < \infty, \|\alpha(y)\xi\| \leq C\|\Lambda_\psi(y)\|, \forall y \in \mathfrak{N}_\psi\}$$

Then, for any ξ in $D({}_\alpha\mathcal{H}, \psi)$, there exists a bounded operator $R^{\alpha, \psi}(\xi)$ from H_ψ to \mathcal{H} , defined, for all y in \mathfrak{N}_ψ by :

$$R^{\alpha, \psi}(\xi)\Lambda_\psi(y) = \alpha(y)\xi$$

This operator belongs to $Hom_N(H_\psi, \mathcal{H})$; therefore, for any ξ, η in $D({}_\alpha\mathcal{H}, \psi)$, the operator :

$$\theta^{\alpha, \psi}(\xi, \eta) = R^{\alpha, \psi}(\xi)R^{\alpha, \psi}(\eta)^*$$

belongs to $\alpha(N)'$; moreover, $D({}_\alpha\mathcal{H}, \psi)$ is dense ([C1], lemma 2), stable under $\alpha(N)'$, and the linear span generated by the operators $\theta^{\alpha, \psi}(\xi, \eta)$ is a weakly dense ideal in $\alpha(N)'$.

With the same hypothesis, the operator :

$$< \xi, \eta >_{\alpha, \psi} = R^{\alpha, \psi}(\eta)^* R^{\alpha, \psi}(\xi)$$

belongs to $\pi_\psi(N)'$. Using Tomita-Takesaki's theory, this last algebra is equal to $J_\psi \pi_\psi(N) J_\psi$, and therefore anti-isomorphic to N (or isomorphic to the opposite von Neumann algebra N^o). We shall consider now $< \xi, \eta >_{\alpha, \psi}$ as an element of N^o , and the linear span generated by these operators is a dense algebra in N^o . More precisely ([C], lemma 4, and [S2], lemme 1.5b), we get that $< \xi, \eta >_{\alpha, \psi}^o$ belongs to \mathfrak{M}_ψ , and that :

$$\Lambda_\psi(< \xi, \eta >_{\alpha, \psi}^o) = J_\psi R^{\alpha, \psi}(\xi)^* \eta$$

If y in N is analytical with respect to ψ , and if $\xi \in D({}_\alpha\mathcal{H}, \psi)$, then we get that $\alpha(y)\xi$ belongs to $D({}_\alpha\mathcal{H}, \psi)$ and that :

$$R^{\alpha, \psi}(\alpha(y)\xi) = R^{\alpha, \psi}(\xi)J_\psi \sigma_{-i/2}^\psi(y^*)J_\psi$$

So, if η is another ψ -bounded element of ${}_\alpha\mathcal{H}$, we get :

$$< \alpha(y)\xi, \eta >_{\alpha, \psi}^o = \sigma_{i/2}^\psi(y) < \xi, \eta >_{\alpha, \psi}^o$$

There exists ([C], prop.3) a family $(e_i)_{i \in I}$ of ψ -bounded elements of ${}_\alpha\mathcal{H}$, such that

$$\sum_i \theta^{\alpha, \psi}(e_i, e_i) = 1$$

Such a family will be called an (α, ψ) -basis of \mathcal{H} .

It is possible ([EN] 2.2) to construct an (α, ψ) -basis of \mathcal{H} , $(e_i)_{i \in I}$, such that the operators $R^{\alpha, \psi}(e_i)$ are partial isometries with final supports $\theta^{\alpha, \psi}(e_i, e_i)$ 2 by 2 orthogonal, and such that, if $i \neq j$, then $< e_i, e_j >_{\alpha, \psi} = 0$. Such a family will be called an (α, ψ) -orthogonal basis of \mathcal{H} .

We have, then :

$$\begin{aligned} R^{\alpha, \psi}(\xi) &= \sum_i \theta^{\alpha, \psi}(e_i, e_i) R^{\alpha, \psi}(\xi) = \sum_i R^{\alpha, \psi}(e_i) < \xi, e_i >_{\alpha, \psi} \\ &< \xi, \eta >_{\alpha, \psi} = \sum_i < \eta, e_i >_{\alpha, \psi}^* < \xi, e_i >_{\alpha, \psi} \end{aligned}$$

$$\xi = \sum_i R^{\alpha, \psi}(e_i) J_\psi \Lambda_\psi(< \xi, e_i >_{\alpha, \psi}^o)$$

the sums being weakly convergent. Moreover, we get that, for all n in N , $\theta^{\alpha, \psi}(e_i, e_i) \alpha(n) e_i = \alpha(n) e_i$, and $\theta^{\alpha, \psi}(e_i, e_i)$ is the orthogonal projection on the closure of the subspace $\{\alpha(n) e_i, n \in N\}$.

If $\theta \in \text{Aut } N$, then it is straightforward to get that $D(\alpha \circ \theta \mathcal{H}, \psi \circ \theta) = D(\alpha \mathcal{H}, \psi)$, and then, we get that, for any ξ, η in $D(\alpha \mathcal{H}, \psi)$:

$$< \xi, \eta >_{\alpha \circ \theta, \psi \circ \theta}^o = \theta^{-1}(< \xi, \eta >_{\alpha, \psi}^o)$$

Let β be a normal non-degenerate anti-representation of N on \mathcal{H} . We may then as well consider \mathcal{H} as a right N -module, and write it \mathcal{H}_β , or consider β as a normal non-degenerate representation of the opposite von Neumann algebra N^o , and consider \mathcal{H} as a left N^o -module.

We can then define on N^o the opposite faithful semi-finite normal weight ψ^o ; we have $\mathfrak{N}_{\psi^o} = \mathfrak{N}_\psi^*$, and the Hilbert space H_{ψ^o} will be, as usual, identified with H_ψ , by the identification, for all x in \mathfrak{N}_ψ , of $\Lambda_{\psi^o}(x^*)$ with $J_\psi \Lambda_\psi(x)$.

From these remarks, we infer that the set of ψ^o -bounded elements of \mathcal{H}_β is :

$$D(\mathcal{H}_\beta, \psi^o) = \{\xi \in \mathcal{H}; \exists C < \infty, \|\beta(y^*)\xi\| \leq C \|\Lambda_\psi(y)\|, \forall y \in \mathfrak{N}_\psi\}$$

and, for any ξ in $D(\mathcal{H}_\beta, \psi^o)$ and y in \mathfrak{N}_ψ , the bounded operator $R^{\beta, \psi^o}(\xi)$ is given by the formula :

$$R^{\beta, \psi^o}(\xi) J_\psi \Lambda_\psi(y) = \beta(y^*) \xi$$

This operator belongs to $\text{Hom}_{N^o}(H_\psi, \mathcal{H})$. Moreover, $D(\mathcal{H}_\beta, \psi^o)$ is dense, stable under $\beta(N)' = P$, and, for all y in P , we have :

$$R^{\beta, \psi^o}(y\xi) = y R^{\beta, \psi^o}(\xi)$$

Then, for any ξ, η in $D(\mathcal{H}_\beta, \psi^o)$, the operator

$$\theta^{\beta, \psi^o}(\xi, \eta) = R^{\beta, \psi^o}(\xi) R^{\beta, \psi^o}(\eta)^*$$

belongs to P , and the linear span generated by these operators is a dense ideal in P ; moreover, the operator-valued product $< \xi, \eta >_{\beta, \psi^o} = R^{\beta, \psi^o}(\eta)^* R^{\beta, \psi^o}(\xi)$ belongs to $\pi_\psi(N)$; we shall consider now, for simplification, that $< \xi, \eta >_{\beta, \psi^o}$ belongs to N , and the linear span generated by these operators is a dense algebra in N , stable under multiplication by analytic elements with respect to ψ . More precisely, $< \xi, \eta >_{\beta, \psi^o}$ belongs to \mathfrak{M}_ψ ([C], lemma 4) and we have ([S1], lemme 1.5)

$$\Lambda_\psi(< \xi, \eta >_{\beta, \psi^o}) = R^{\beta, \psi^o}(\eta)^* \xi$$

A (β, ψ^o) -basis of \mathcal{H} is a family $(e_i)_{i \in I}$ of ψ^o -bounded elements of \mathcal{H}_β , such that

$$\sum_i \theta^{\beta, \psi^o}(e_i, e_i) = 1$$

We have then, for all ξ in $D(\mathcal{H}_\beta)$:

$$\xi = \sum_i R^{\beta, \psi^o}(e_i) \Lambda_\psi(< \xi, e_i >_{\beta, \psi^o})$$

It is possible to choose the $(e_i)_{i \in I}$ such that the $R^{\beta, \psi^o}(e_i)$ are partial isometries, with final supports $\theta^{\beta, \psi^o}(e_i, e_i)$ 2 by 2 orthogonal, and such that $< e_i, e_j >_{\beta, \psi^o} = 0$ if $i \neq j$; such a family will be then called a (β, ψ^o) -orthogonal basis of \mathcal{H} . We have then

$$R^{\beta, \psi^o}(e_i) = \theta^{\beta, \psi^o}(e_i, e_i) R^{\beta, \psi^o}(e_i) = R^{\beta, \psi^o}(e_i) < e_i, e_i >_{\beta, \psi^o}$$

Moreover, we get that, for all n in N , and for all i , we have :

$$\theta^{\beta, \psi^o}(e_i, e_i) \beta(n) e_i = \beta(n) e_i$$

and that $\theta^{\beta, \psi^o}(e_i, e_i)$ is the orthogonal projection on the closure of the subspace $\{\beta(n) e_i, n \in N\}$.

2.2. Jones' basic construction and operator-valued weights.

Let $M_0 \subset M_1$ be an inclusion of von Neumann algebras (for simplification, these algebras will be supposed to be σ -finite), equipped with a normal faithful semi-finite operator-valued weight T_1 from M_1 to M_0 (to be more precise, from M_1^+ to the extended positive elements of M_0 (cf. [T] IX.4.12)). Let ψ_0 be a normal faithful semi-finite weight on M_0 , and $\psi_1 = \psi_0 \circ T_1$; for $i = 0, 1$, let $H_i = H_{\psi_i}$, $J_i = J_{\psi_i}$, $\Delta_i = \Delta_{\psi_i}$ be the usual objects constructed by the Tomita-Takesaki theory associated to these weights. Following ([J], 3.1.5(i)), the von Neumann algebra $M_2 = J_1 M_0' J_1$ defined on the Hilbert space H_1 will be called the basic construction made from the inclusion $M_0 \subset M_1$. We have $M_1 \subset M_2$, and we shall say that the inclusion $M_0 \subset M_1 \subset M_2$ is standard. Following ([EN] 10.6), for x in \mathfrak{N}_{T_1} , we shall define $\Lambda_{T_1}(x)$ by the following formula, for all z in \mathfrak{N}_{ψ_0} :

$$\Lambda_{T_1}(x) \Lambda_{\psi_0}(z) = \Lambda_{\psi_1}(xz)$$

Then, $\Lambda_{T_1}(x)$ belongs to $\text{Hom}_{M_0^o}(H_0, H_1)$; if x, y belong to \mathfrak{N}_{T_1} , then $\Lambda_{T_1}(x)^* \Lambda_{T_1}(y) = T_1(x^* y)$, and $\Lambda_{T_1}(x) \Lambda_{T_1}(y)^*$ belongs to M_2 .

Using then Haagerup's construction ([T], IX.4.24), it is possible to construct a normal semi-finite faithful operator-valued weight T_2 from M_2 to M_1 ([EN], 10.7), which will be called the basic construction made from T_1 . If x, y belong to \mathfrak{N}_{T_1} , then $\Lambda_{T_1}(x) \Lambda_{T_1}(y)^*$ belongs to \mathfrak{M}_{T_2} , and $T_2(\Lambda_{T_1}(x) \Lambda_{T_1}(y)^*) = xy^*$.

By Tomita-Takesaki theory, the Hilbert space H_1 bears a natural structure of $M_1 - M_1^o$ -bimodule, and, therefore, by restriction, of $M_0 - M_0^o$ -bimodule. Let us write r for the canonical representation of M_0 on H_1 , and s for the canonical antirepresentation given, for all x in M_0 , by $s(x) = J_1 r(x)^* J_1$. Let us have now a closer look to the subspaces

$D(H_{1s}, \psi_0^o)$ and $D(rH_1, \psi_0)$. If x belongs to $\mathfrak{N}_{T_1} \cap \mathfrak{N}_{\psi_1}$, we easily get that $J_1 \Lambda_{\psi_1}(x)$ belongs to $D(rH_1, \psi_0)$, with :

$$R^{r, \psi_0}(J_1 \Lambda_{\psi_1}(x)) = J_1 \Lambda_{T_1}(x) J_0$$

and $\Lambda_{\psi_1}(x)$ belongs to $D(H_{1s}, \psi_0)$, with :

$$R^{s, \psi_0^o}(\Lambda_{\psi_1}(x)) = \Lambda_{T_1}(x)$$

In ([E3], 2.3) was proved that the subspace $D(H_{1s}, \psi_0^o) \cap D(rH_1, \psi_0)$ is dense in H_1 ; let us write down and precise this result :

2.2.1. Proposition. *Let us keep on the notations of this paragraph; let $\mathcal{T}_{\psi_1, T_1}$ be the algebra made of elements x in $\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_1} \cap \mathfrak{N}_{\psi_1}^* \cap \mathfrak{N}_{T_1}^*$, analytical with respect to ψ_1 , and such that, for all z in \mathbb{C} , $\sigma_z^{\psi_1}(x)$ belongs to $\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_1} \cap \mathfrak{N}_{\psi_1}^* \cap \mathfrak{N}_{T_1}^*$. Then :*

- (i) *the algebra $\mathcal{T}_{\psi_1, T_1}$ is weakly dense in M_1 ; it will be called Tomita's algebra with respect to ψ_1 and T_1 ;*
- (ii) *for any x in $\mathcal{T}_{\psi_1, T_1}$, $\Lambda_{\psi_1}(x)$ belongs to $D(H_{1s}, \psi_0^o) \cap D(rH_1, \psi_0)$;*
- (iii) *for any ξ in $D(H_{1s}, \psi_0^o)$, there exists a sequence x_n in $\mathcal{T}_{\psi_1, T_1}$ such that $\Lambda_{T_1}(x_n) = R^{s, \psi_0^o}(\Lambda_{\psi_1}(x_n))$ is weakly converging to $R^{s, \psi_0^o}(\xi)$ and $\Lambda_{\psi_1}(x_n)$ is converging to ξ .*

Proof. The result (i) is taken from ([EN], 10.12); we get in ([E3], 2.3) an increasing sequence of projections p_n in M_1 , converging to 1, and elements x_n in $\mathcal{T}_{\psi_1, T_1}$ such that $\Lambda_{\psi_1}(x_n) = p_n \xi$. So, (i) and (ii) were obtained in ([E3], 2.3) from this construction. More precisely, we get that :

$$\begin{aligned} T_1(x_n^* x_n) &= \langle R^{s, \psi_0^o}(\Lambda_{\psi_1}(x_n)), R^{s, \psi_0^o}(\Lambda_{\psi_1}(x_n)) \rangle_{s, \psi_0^o} \\ &= \langle p_n \xi, p_n \xi \rangle_{s, \psi_0^o} \\ &= R^{s, \psi_0^o}(\xi)^* p_n R^{s, \psi_0^o}(\xi) \end{aligned}$$

which is increasing and weakly converging to $\langle \xi, \xi \rangle_{s, \psi_0^o}$. \square

We finish by writing a proof of this useful lemma, we were not able to find in litterature :

2.2.2. Lemma. *Let $M_0 \subset M_1$ be an inclusion of von neumann algebras, equipped with a normal faithful semi-finite operator-valued weight T from M_1 to M_0 . Let ψ_0 be a normal semi-finite faithful weight on M_0 , and $\psi_1 = \psi_0 \circ T$; if x is in \mathfrak{N}_T , and if y is in $M_0' \cap M_1$, analytical with respect to ψ_1 , then xy belongs to \mathfrak{N}_T .*

Proof. Let a be in \mathfrak{N}_{ψ_0} ; then xa belongs to \mathfrak{N}_{ψ_1} , and $xya = xay$ belongs to \mathfrak{N}_{ψ_1} ; moreover, let us consider the element $T(y^* x^* xy)$ of the positive extended part of M_0^+ ; we have :

$$\begin{aligned} &\langle T(y^* x^* xy), \omega_{\Lambda_{\psi_0}}(a) \rangle = \psi_1(a^* y^* x^* xya) = \|\Lambda_{\psi_1}(xay)\|^2 = \\ &= \|J_{\psi_1} \sigma_{-i/2}^{\psi_1}(y^*) J_{\psi_1} \Lambda_{\psi_1}(xa)\|^2 = \|J_{\psi_1} \sigma_{-i/2}^{\psi_1}(y^*) J_{\psi_1} \Lambda_T(x) \Lambda_{\psi_0}(a)\|^2 \end{aligned}$$

from which we get that $T(y^*x^*xy)$ is bounded and

$$T(y^*x^*xy) \leq \|\sigma_{-i/2}^{\psi_1}(y^*)\|^2 T(x^*x)$$

□

2.3. Relative tensor product [C1], [S2], [T]. Using the notations of 2.1, let now \mathcal{K} be another Hilbert space on which there exists a non-degenerate representation γ of N . Following J.-L. Sauvageot ([S2], 2.1), we define the relative tensor product $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ as the Hilbert space obtained from the algebraic tensor product $D(\mathcal{H}_\beta, \psi^o) \odot \mathcal{K}$ equipped with the scalar product defined, for ξ_1, ξ_2 in $D(\mathcal{H}_\beta, \psi^o)$, η_1, η_2 in \mathcal{K} , by

$$(\xi_1 \odot \eta_1 | \xi_2 \odot \eta_2) = (\gamma(< \xi_1, \xi_2 >_{\beta, \psi^o}) \eta_1 | \eta_2)$$

where we have identified N with $\pi_\psi(N)$ to simplify the notations. The image of $\xi \odot \eta$ in $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ will be denoted by $\xi_{\beta \otimes_\gamma \eta}$. We shall use intensively this construction; one should bear in mind that, if we start from another faithful semi-finite normal weight ψ' , we get another Hilbert space $\mathcal{H}_{\beta \otimes_{\gamma'} \mathcal{K}}$; there exists an isomorphism $U_{\beta, \gamma}^{\psi, \psi'}$ from $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ to $\mathcal{H}_{\beta \otimes_{\gamma'} \mathcal{K}}$, which is unique up to some functorial property ([S2], 2.6) (but this isomorphism does not send $\xi_{\beta \otimes_\gamma \eta}$ on $\xi_{\beta \otimes_{\gamma'} \eta}$!).

When no confusion is possible about the representation and the anti-representation, we shall write $\mathcal{H} \otimes_\psi \mathcal{K}$ instead of $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$, and $\xi \otimes_\psi \eta$ instead of $\xi_{\beta \otimes_\gamma \eta}$.

If $\theta \in \text{Aut } N$, then, using a remark made in 2.1, we get that the application which sends $\xi_{\beta \otimes_\gamma \eta}$ onto $\xi_{\beta \circ \theta \otimes_{\alpha \circ \theta} \eta}$ leads to a unitary from $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ onto $\mathcal{H}_{\beta \circ \theta \otimes_{\alpha \circ \theta} \mathcal{K}}$.

For any ξ in $D(\mathcal{H}_\beta, \psi^o)$, we define the bounded linear application $\lambda_\xi^{\beta, \gamma}$ from \mathcal{K} to $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ by, for all η in \mathcal{K} , $\lambda_\xi^{\beta, \gamma}(\eta) = \xi_{\beta \otimes_\gamma \eta}$. We shall write λ_ξ if no confusion is possible. We get ([EN], 3.10) :

$$\lambda_\xi^{\beta, \gamma} = R^{\beta, \psi^o}(\xi) \otimes_\psi 1_{\mathcal{K}}$$

where we recall the canonical identification (as left N -modules) of $L^2(N) \otimes_\psi \mathcal{K}$ with \mathcal{K} . We have :

$$(\lambda_\xi^{\beta, \gamma})^* \lambda_\xi^{\beta, \gamma} = \gamma(< \xi, \xi >_{\beta, \psi^o})$$

In ([S1] 2.1), the relative tensor product $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ is defined also, if ξ_1 ,

ξ_2 are in \mathcal{H} , η_1, η_2 are in $D(\gamma\mathcal{K}, \psi)$, by the following formula :

$$(\xi_1 \odot \eta_1 | \xi_2 \odot \eta_2) = (\beta(< \eta_1, \eta_2 >_{\gamma, \psi}) \xi_1 | \xi_2)$$

which leads to the the definition of a relative flip σ_ψ which will be an isomorphism from $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ onto $\mathcal{K}_{\gamma \otimes_\beta \mathcal{H}}$, defined, for any ξ in $D(\mathcal{H}_\beta, \psi^\circ)$, η in $D(\gamma\mathcal{K}, \psi)$, by :

$$\sigma_\psi(\xi \otimes_\psi \eta) = \eta \otimes_{\psi^\circ} \xi$$

This allows us to define a relative flip ς_ψ from $\mathcal{L}(\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}})$ to $\mathcal{L}(\mathcal{K}_{\gamma \otimes_\beta \mathcal{H}})$ which sends X in $\mathcal{L}(\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}})$ onto $\varsigma_\psi(X) = \sigma_\psi X \sigma_\psi^*$. Starting from another faithful semi-finite normal weight ψ' , we get a von Neumann algebra $\mathcal{L}(\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}})$ which is isomorphic to $\mathcal{L}(\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}})$, and a von Neumann algebra $\mathcal{L}(\mathcal{K}_{\gamma \otimes_\beta \mathcal{H}})$ which is isomorphic to $\mathcal{L}(\mathcal{K}_{\gamma \otimes_\beta \mathcal{H}})$; as we get that :

$$\sigma_{\psi'} \circ U_{\beta, \gamma}^{\psi, \psi'} = U_{\gamma, \beta}^{\psi^\circ, \psi'^\circ}$$

we see that these isomorphisms exchange ς_ψ and $\varsigma_{\psi'}$. Therefore, the homomorphism ς_ψ can be denoted ς_N without any reference to a specific weight.

We may define, for any η in $D(\gamma\mathcal{K}, \psi)$, an application $\rho_\eta^{\beta, \gamma}$ from \mathcal{H} to $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$ by $\rho_\eta^{\beta, \gamma}(\xi) = \xi_{\beta \otimes_\gamma \eta}$. We shall write ρ_η if no confusion is possible. We get that :

$$(\rho_\eta^{\beta, \gamma})^* \rho_\eta^{\beta, \gamma} = \beta(< \eta, \eta >_{\gamma, \psi})$$

We recall, following ([S2], 2.2b) that, for all ξ in \mathcal{H} , η in $D(\gamma\mathcal{K}, \psi)$, y in N , analytic with respect to ψ , we have :

$$\beta(y)\xi \otimes_\psi \eta = \xi \otimes_\psi \gamma(\sigma_{-i/2}^\psi(y))\eta$$

Let x be an element of $\mathcal{L}(\mathcal{H})$, commuting with the right action of N on \mathcal{H}_β (i.e. $x \in \beta(N)'$). It is possible to define an operator $x_{\beta \otimes_\gamma 1_\mathcal{K}}$ on $\mathcal{H}_{\beta \otimes_\gamma \mathcal{K}}$. We can easily evaluate $\|x_{\beta \otimes_\gamma 1_\mathcal{K}}\|$, for any finite $J \subset I$, for any η_i in \mathcal{K} , we have :

$$\begin{aligned} ((x^* x_{\beta \otimes_\gamma 1_\mathcal{K}})(\sum_{i \in J} e_i_{\beta \otimes_\gamma \eta_i}) | (\sum_{i \in J} e_i_{\beta \otimes_\gamma \eta_i})) &= \\ &= \sum_{i \in J} (\gamma(< x e_i, x e_i >_{\beta, \psi^\circ}) \eta_i | \eta_i) \\ &\leq \|x\|^2 \sum_{i \in J} (\gamma(< e_i, e_i >_{\beta, \psi^\circ}) \eta_i | \eta_i) = \|x\|^2 \|\sum_{i \in J} e_i_{\beta \otimes_\gamma \eta_i}\| \end{aligned}$$

from which we get $\|x_{\beta \otimes_\gamma 1_\mathcal{K}}\| \leq \|x\|$.

By the same way, if y commutes with the left action of N on $\gamma\mathcal{K}$ (i.e. y

is in $\gamma(N)'$, it is possible to define $1_{\mathcal{H}_{\beta \otimes_{\gamma} y}}$ on $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$, and by composition, it is possible to define then $x_{\beta \otimes_{\gamma} y}$. If we start from another faithful semi-finite normal weight ψ' , the canonical isomorphism $U_{\beta, \gamma}^{\psi, \psi'}$ from $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$ to $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$ sends $x_{\beta \otimes_{\gamma} y}$ on $x_{\beta \otimes_{\gamma} y}$ ([S2], 2.3 and 2.6); therefore, this operator can be denoted $x_{\beta \otimes_{\gamma} y}$ without any reference to a specific weight, and we get $\|x_{\beta \otimes_{\gamma} y}\| \leq \|x\| \|y\|$. If $\theta \in \text{Aut} N$, the unitary from $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$ onto $\mathcal{H}_{\beta \circ \theta \otimes_{\alpha \circ \theta} \mathcal{K}}$ sends $x_{\beta \otimes_{\gamma} y}$ on $x_{\beta \circ \theta \otimes_{\gamma \circ \theta} y}$. With the notations of 2.1, let $(e_i)_{i \in I}$ a (β, ψ^o) -orthogonal basis of \mathcal{H} ; let us remark that, for all η in \mathcal{K} , we have :

$$e_i \beta \otimes_{\gamma} \eta = e_i \beta \otimes_{\gamma} \gamma(< e_i, e_i >_{\beta, \psi^o}) \eta$$

On the other hand, $\theta^{\beta, \psi^o}(e_i, e_i)$ is an orthogonal projection, and so is $\theta^{\beta, \psi^o}(e_i, e_i) \beta \otimes_{\gamma} 1$; this last operator is the projection on the subspace $e_i \beta \otimes_{\gamma} \gamma(< e_i, e_i >_{\beta, \psi^o}) \mathcal{K}$ ([E2], 2.3) and, therefore, we get that $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$ is the orthogonal sum of the subspaces $e_i \beta \otimes_{\gamma} \gamma(< e_i, e_i >_{\beta, \psi^o}) \mathcal{K}$; for any Ξ in $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$, there exist ξ_i in \mathcal{K} , such that $\gamma(< e_i, e_i >_{\beta, \psi^o}) \xi_i = \xi_i$ and $\Xi = \sum_i e_i \beta \otimes_{\gamma} \xi_i$, from which we get that $\sum_i \|\xi_i\|^2 = \|\Xi\|^2$.

Let us suppose now that \mathcal{K} is a $N - P$ bimodule; that means that there exists a von Neumann algebra P , and a non-degenerate normal anti-representation ϵ of P on \mathcal{K} , such that $\epsilon(P) \subset \gamma(N)'$. We shall write then ${}_{\gamma} \mathcal{K}_{\epsilon}$. If y is in P , we have seen that it is possible to define then the operator $1_{\mathcal{H}_{\beta \otimes_{\gamma} \epsilon(y)}}$ on $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$, and we define this way a non-degenerate normal antirepresentation of P on $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$, we shall call again ϵ for simplification. If \mathcal{H} is a $Q - N$ bimodule, then $\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}}$ becomes a $Q - P$ bimodule (Connes' fusion of bimodules).

Taking a faithful semi-finite normal weight ν on P , and a left P -module ${}_{\zeta} \mathcal{L}$ (i.e. a Hilbert space \mathcal{L} and a normal non-degenerate representation ζ of P on \mathcal{L}), it is possible then to define $(\mathcal{H}_{\beta \otimes_{\gamma} \mathcal{K}})_{\epsilon \otimes_{\zeta} \mathcal{L}}$. Of course, it is possible also to consider the Hilbert space $\mathcal{H}_{\beta \otimes_{\gamma} (\mathcal{K}_{\epsilon \otimes_{\zeta} \mathcal{L}})}$. It can be shown that these two Hilbert spaces are isomorphics as $\beta(N)' - \zeta(P)'$ -bimodules. (In ([V1] 2.1.3), the proof, given for $N = P$ abelian can be used, without modification, in that wider hypothesis). We shall write

then $\mathcal{H}_{\beta \otimes_{\psi} \gamma} \mathcal{K}_{\epsilon \otimes_{\nu} \zeta} \mathcal{L}$ without parenthesis, to emphasise this coassociativity property of the relative tensor product.

Dealing now with that Hilbert space $\mathcal{H}_{\beta \otimes_{\psi} \gamma} \mathcal{K}_{\epsilon \otimes_{\nu} \zeta} \mathcal{L}$, there exist different flips, and it is necessary to be careful with notations. For instance, $1_{\beta} \otimes \sigma_{\nu}$ is the flip from this Hilbert space onto $\mathcal{H}_{\beta \otimes_{\psi} \gamma} (\mathcal{L}_{\zeta \otimes_{\nu^o} \epsilon} \mathcal{K})$, where γ is here acting on the second leg of $\mathcal{L}_{\zeta \otimes_{\nu^o} \epsilon} \mathcal{K}$ (and should therefore be written $1_{\zeta \otimes_{\nu^o} \epsilon} \gamma$, but this will not be done for obvious reasons). Here, the parenthesis remains, because there is no associativity rule, and to remind that γ is not acting on \mathcal{L} . The adjoint of $1_{\beta} \otimes \sigma_{\nu}$ is $1_{\beta} \otimes \sigma_{\nu^o}$.

The same way, we can consider $\sigma_{\psi} \epsilon \otimes_{\nu} 1$ from $\mathcal{H}_{\beta \otimes_{\psi} \gamma} \mathcal{K}_{\epsilon \otimes_{\nu} \zeta} \mathcal{L}$ onto $(\mathcal{K}_{\gamma \otimes_{\psi^o} \beta} \mathcal{H})_{\epsilon \otimes_{\nu} \zeta} \mathcal{L}$.

Another kind of flip sends $\mathcal{H}_{\beta \otimes_{\psi} \gamma} (\mathcal{L}_{\zeta \otimes_{\nu^o} \epsilon} \mathcal{K})$ onto $\mathcal{L}_{\zeta \otimes_{\nu^o} \epsilon} (\mathcal{H}_{\beta \otimes_{\psi} \gamma} \mathcal{K})$.

We shall denote this application $\sigma_{\gamma, \epsilon}^{1,2}$ (and its adjoint $\sigma_{\epsilon, \gamma}^{1,2}$), in order to emphasize that we are exchanging the first and the second leg, and the representation γ and ϵ on the third leg.

If π denotes the canonical left representation of N on the Hilbert space $L^2(N)$, then it is straightforward to verify that the application which sends, for all ξ in \mathcal{H} , χ normal faithful semi-finite weight on N , and x in \mathfrak{N}_{χ} , the vector $\xi_{\beta} \otimes_{\pi} J_{\chi} \Lambda_{\chi}(x)$ on $\beta(x^*)\xi$, gives an isomorphism of $\mathcal{H}_{\beta} \otimes_{\pi} L^2(N)$ on \mathcal{H} , which will send the antirepresentation of N given by $n \mapsto 1_{\mathcal{H}\beta} \otimes_{\pi} J_{\chi} n^* J_{\chi}$ on β

If \mathcal{K} is a Hilbert space on which there exists a non-degenerate representation γ of N , then \mathcal{K} is a $N - \gamma(N)^{o}$ bimodule, and the conjugate Hilbert space $\overline{\mathcal{K}}$ is a $\gamma(N)' - N^o$ bimodule, and, ([S2]), for any normal faithful semi-finite weight ϕ on $\gamma(N)'$, the fusion ${}_{\gamma} \mathcal{K} \otimes_{\phi^o} \overline{\mathcal{K}}_{\gamma}$ is isomorphic to the standard space $L^2(N)$, equipped with its standard left and right representation.

Using that remark, one gets for any $x \in \beta(N)'$:

$$\|x_{\beta \otimes_{\gamma} 1_{\mathcal{K}}}\| \leq \|x_{\beta \otimes_{\gamma} 1_{\mathcal{K}}} \otimes_{\gamma(N)^{o}} 1_{\overline{\mathcal{K}}}\| = \|x_{\beta \otimes 1_{L^2(N)}}\| = \|x\|$$

from which we have $\|x_{\beta \otimes_{\gamma} 1_{\mathcal{K}}}\| = \|x\|$.

2.4. Fiber product [V1], [EV]. Let us follow the notations of 2.3; let now M_1 be a von Neumann algebra on \mathcal{H} , such that $\beta(N) \subset M_1$, and M_2 be a von Neumann algebra on \mathcal{K} , such that $\gamma(N) \subset M_2$. The von Neumann algebra generated by all elements $x_{\beta \otimes_{\gamma} y}$, where x belongs

to M'_1 , and y belongs M'_2 will be denoted $M'_1 \underset{N}{\beta \otimes_\gamma} M'_2$ (or $M'_1 \otimes_N M'_2$ if no confusion is possible), and will be called the relative tensor product of M'_1 and M'_2 over N . The commutant of this algebra will be denoted $M_1 \underset{N}{\beta *_\gamma} M_2$ (or $M_1 *_N M_2$ if no confusion is possible) and called the fiber product of M_1 and M_2 , over N . If $\theta \in \text{Aut} N$, using a remark made in 2.3, we get that the von Neumann algebras $M_1 \underset{N}{\beta *_\gamma} M_2$ and $M_1 \underset{N}{\beta \circ \theta *_{\gamma \circ \theta}} M_2$ are spatially isomorphic, and we shall identify them.

It is straightforward to verify that, if P_1 and P_2 are two other von Neumann algebras satisfying the same relations with N , we have

$$M_1 *_N M_2 \cap P_1 *_N P_2 = (M_1 \cap P_1) *_N (M_2 \cap P_2)$$

Moreover, we get that $\varsigma_N(M_1 \underset{N}{\beta *_\gamma} M_2) = M_2 \underset{N^o}{\gamma *_\beta} M_1$.

In particular, we have :

$$(M_1 \cap \beta(N)') \underset{N}{\beta \otimes_\gamma} (M_2 \cap \gamma(N)') \subset M_1 \underset{N}{\beta *_\gamma} M_2$$

and :

$$M_1 \underset{N}{\beta *_\gamma} \gamma(N) = (M_1 \cap \beta(N)') \underset{N}{\beta \otimes_\gamma} 1$$

More generally, if β is a non-degenerate normal involutive antihomomorphism from N into a von Neumann algebra M_1 , and γ a non-degenerate normal involutive homomorphism from N into a von Neumann algebra M_2 , it is possible to define, without any reference to a specific Hilbert space, a von Neumann algebra $M_1 \underset{N}{\beta *_\gamma} M_2$.

Moreover, if now β' is a non-degenerate normal involutive antihomomorphism from N into another von Neumann algebra P_1 , γ' a non-degenerate normal involutive homomorphism from N into another von Neumann algebra P_2 , Φ a normal involutive homomorphism from M_1 into P_1 such that $\Phi \circ \beta = \beta'$, and Ψ a normal involutive homomorphism from M_2 into P_2 such that $\Psi \circ \gamma = \gamma'$, it is possible then to define a normal involutive homomorphism (the proof given in ([S1] 1.2.4) in the case when N is abelian can be extended without modification in the general case) :

$$\Phi \underset{N}{\beta *_\gamma} \Psi : M_1 \underset{N}{\beta *_\gamma} M_2 \mapsto P_1 \underset{N}{\beta' *_{\gamma'}} P_2$$

Let Φ be in $\text{Aut} M_1$, Ψ in $\text{Aut} M_2$, and let $\theta \in \text{Aut} N$ be such that $\Phi \circ \beta = \beta \circ \theta$ and $\Psi \circ \gamma = \gamma \circ \theta$, then, using the identification between $M_1 \underset{N}{\beta *_\gamma} M_2$ and $M_1 \underset{N}{\beta \circ \theta *_{\gamma \circ \theta}} M_2$, we get the existence of an automorphism $\Phi \underset{N}{\beta *_\gamma} \Psi$ of $M_1 \underset{N}{\beta *_\gamma} M_2$.

In the case when ${}_\gamma \mathcal{K}_\epsilon$ is a $N - P^o$ bimodule as explained in 2.3 and ${}_\zeta \mathcal{L}$ a P -module, if $\gamma(N) \subset M_2$ and $\epsilon(P) \subset M_2$, and if $\zeta(P) \subset M_3$, where M_3 is a von Neumann algebra on \mathcal{L} , it is possible to consider

then $(M_1 \underset{N}{\beta} *_{\gamma} M_2) \underset{P}{\epsilon} *_{\zeta} M_3$ and $M_1 \underset{N}{\beta} *_{\gamma} (M_2 \underset{P}{\epsilon} *_{\zeta} M_3)$. The coassociativity property for relative tensor products leads then to the isomorphism of these von Neumann algebra we shall write now $M_1 \underset{N}{\beta} *_{\gamma} M_2 \underset{P}{\epsilon} *_{\zeta} M_3$ without parenthesis.

2.5. Slice maps [E3]. Let A be in $M_1 \underset{N}{\beta} *_{\gamma} M_2$, ψ a normal faithful semi-finite weight on N , \mathcal{H} an Hilbert space on which M_1 is acting, \mathcal{K} an Hilbert space on which M_2 is acting, and let ξ_1, ξ_2 be in $D(\mathcal{H}_{\beta}, \psi^o)$; let us define :

$$(\omega_{\xi_1, \xi_2} \underset{\psi}{\beta} *_{\gamma} id)(A) = (\lambda_{\xi_2}^{\beta, \gamma})^* A \lambda_{\xi_1}^{\beta, \gamma}$$

We define this way $(\omega_{\xi_1, \xi_2} \underset{\psi}{\beta} *_{\gamma} id)(A)$ as a bounded operator on \mathcal{K} , which belongs to M_2 , such that :

$$((\omega_{\xi_1, \xi_2} \underset{\psi}{\beta} *_{\gamma} id)(A) \eta_1 | \eta_2) = (A(\xi_1 \underset{\psi}{\beta} \otimes_{\gamma} \eta_1) | \xi_2 \underset{\psi}{\beta} \otimes_{\gamma} \eta_2)$$

One should note that $(\omega_{\xi_1, \xi_2} \underset{\psi}{\beta} *_{\gamma} id)(1) = \gamma(< \xi_1, \xi_2 >_{\beta, \psi^o})$.

Let us define the same way, for any η_1, η_2 in $D(\gamma \mathcal{K}, \psi)$:

$$(id \underset{\psi}{\beta} *_{\gamma} \omega_{\eta_1, \eta_2})(A) = (\rho_{\eta_2}^{\beta, \gamma})^* A \rho_{\eta_1}^{\beta, \gamma}$$

which belongs to M_1 .

We therefore have a Fubini formula for these slice maps : for any ξ_1, ξ_2 in $D(\mathcal{H}_{\beta}, \psi^o)$, η_1, η_2 in $D(\gamma \mathcal{K}, \psi)$, we have :

$$< (\omega_{\xi_1, \xi_2} \underset{\psi}{\beta} *_{\gamma} id)(A), \omega_{\eta_1, \eta_2} > = < (id \underset{\psi}{\beta} *_{\gamma} \omega_{\eta_1, \eta_2})(A), \omega_{\xi_1, \xi_2} >$$

Let ϕ_1 be a normal semi-finite weight on M_1^+ , and A be a positive element of the fiber product $M_1 \underset{N}{\beta} *_{\gamma} M_2$, then we may define an element of the extended positive part of M_2 , denoted $(\phi_1 \underset{\psi}{\beta} *_{\gamma} id)(A)$, such that, for all η in $D(\gamma L^2(M_2), \psi)$, we have :

$$\|(\phi_1 \underset{\psi}{\beta} *_{\gamma} id)(A)^{1/2} \eta\|^2 = \phi_1(id \underset{\psi}{\beta} *_{\gamma} \omega_{\eta})(A)$$

Moreover, then, if ϕ_2 is a normal semi-finite weight on M_2^+ , we have :

$$\phi_2(\phi_1 \underset{\psi}{\beta} *_{\gamma} id)(A) = \phi_1(id \underset{\psi}{\beta} *_{\gamma} \phi_2)(A)$$

and if ω_i are in M_{1*} such that $\phi_1 = \sup_i \omega_i$, we have $(\phi_1 \underset{\psi}{\beta} *_{\gamma} id)(A) = \sup_i (\omega_i \underset{\psi}{\beta} *_{\gamma} id)(A)$.

Let now P_1 be a von Neuman algebra such that :

$$\beta(N) \subset P_1 \subset M_1$$

and let Φ_i ($i = 1, 2$) be a normal faithful semi-finite operator valued weight from M_i to P_i ; for any positive operator A in the fiber product $M_1 \beta^* \gamma_N M_2$, there exists an element $(\Phi_1 \beta^* \gamma_N id)(A)$ of the extended positive part of $P_1 \beta^* \gamma_N M_2$, such that ([E3], 3.5), for all η in $D(\gamma L^2(M_2), \psi)$, and ξ in $D(L^2(P_1)_\beta, \psi^o)$, we have :

$$\|(\Phi_1 \beta^* \gamma_N id)(A)^{1/2}(\xi \beta \otimes_\gamma \eta)\|_\psi^2 = \|\Phi_1(id \beta^* \gamma_N \omega_\eta)(A)^{1/2}\xi\|_\psi^2$$

If ϕ is a normal semi-finite weight on P , we have :

$$(\phi \circ \Phi_1 \beta^* \gamma_N id)(A) = (\phi \beta^* \gamma_N id)(\Phi_1 \beta^* \gamma_N id)(A)$$

We define the same way an element $(id \beta^* \gamma_N \Phi_2)(A)$ of the extended positive part of $M_1 \gamma^* \beta_N P_2$, and we have :

$$(id \beta^* \gamma_N \Phi_2)((\Phi_1 \beta^* \gamma_N id)(A)) = (\Phi_1 \beta^* \gamma_N id)((id \beta^* \gamma_N \Phi_2)(A))$$

Considering now an element x of $M_1 \beta^* \pi_\psi(N)$, which can be identified (2.4) to $M_1 \cap \beta(N)'$, we get that, for e in \mathfrak{N}_ψ , we have

$$(id_\beta * \pi_\psi \omega_{J_\psi \Lambda_\psi(e)})(x) = \beta(ee^*)x$$

Therefore, by increasing limits, we get that $(id_\beta * \pi_\psi \psi)$ is the injection of $M_1 \cap \beta(N)'$ into M_1 . More precisely, if x belongs to $M_1 \cap \beta(N)'$, we have :

$$(id_\beta * \pi_\psi \psi)(x_\beta \otimes_\psi \pi 1) = x$$

Therefore, if Φ_2 is a normal faithful semi-finite operator-valued weight from M_2 onto $\gamma(N)$, we get that, for all A positive in $M_1 \beta^* \gamma_N M_2$, we have :

$$(id_\beta * \gamma \psi \circ \Phi_2)(A)_\beta \otimes_\gamma 1 = (id_\beta * \gamma \Phi_2)(A)$$

With the notations of 2.1, let $(e_i)_{i \in I}$ be a (β, ψ^o) -orthogonal basis of \mathcal{H} ; using the fact (2.3) that, for all η in \mathcal{K} , we have :

$$e_i \beta \otimes_\gamma \eta = e_i \beta \otimes_\gamma \gamma(< e_i, e_i >_{\beta, \psi^o}) \eta$$

we get that, for all X in $M_1 \beta^* \gamma_N M_2$, ξ in $D(\mathcal{H}_\beta, \psi^o)$, we have

$$(\omega_{\xi, e_i \beta^* \gamma_N id})(X) = \gamma(< e_i, e_i >_{\beta, \psi^o})(\omega_{\xi, e_i \beta^* \gamma_N id})(X)$$

2.6. Vaes' Radon-Nikodym theorem. In [V] is proved a very nice Radon-Nikodym theorem for two normal faithful semi-finite weights on a von Neumann algebra M . If Φ and Ψ are such weights, then are equivalent :

- the two modular automorphism groups σ^Φ and σ^Ψ commute;
- the Connes' derivative $[D\Psi : D\Phi]_t$ is of the form :

$$[D\Psi : D\Phi]_t = \lambda^{it^2/2} \delta^{it}$$

where λ is a non-singular positive operator affiliated to $Z(M)$, and δ is a non-singular positive operator affiliated to M .

It is then easy to verify that $\sigma_t^\Phi(\delta^{is}) = \lambda^{ist} \delta^{is}$, and that

$$[D\Phi \circ \sigma_t^\Psi : D\Phi]_s = \lambda^{ist}$$

$$[D\Psi \circ \sigma_t^\Phi : D\Psi]_s = \lambda^{-ist}$$

Moreover, we have also, for any $x \in M^+$:

$$\Psi(x) = \lim_n \Phi((\delta^{1/2} e_n) x (\delta^{1/2} e_n))$$

where the e_n are self-adjoint elements of M given by the formula :

$$e_n = a_n \int_{\mathbb{R}^2} e^{-n^2 x^2 - n^4 y^4} \lambda^{ix} \delta^{iy} dx dy$$

where we put $a_n = 2n^2 \Gamma(1/2)^{-1} \Gamma(1/4)^{-1}$. The operators e_n are analytic with respect to σ^Φ and such that, for any $z \in \mathbb{C}$, the sequence $\sigma_z^\Phi(e_n)$ is bounded and strongly converges to 1.

In that situation, we shall write $\Psi = \Phi_\delta$ and call δ the modulus of Ψ with respect to Φ ; λ will be called the scaling operator of Ψ with respect to Φ .

Moreover, if $a \in M$ is such that $a\delta^{1/2}$ is bounded and its closure $\overline{a\delta^{1/2}}$ belongs to \mathfrak{N}_Φ , then a belongs to \mathfrak{N}_Ψ . We may then identify $\Lambda_\Psi(a)$ with $\Lambda_\Phi(\overline{a\delta^{1/2}})$, J_Ψ with $\lambda^{i/4} J_\Phi$, Δ_Ψ with $\overline{J_\Phi \delta^{-1} J_\Phi \delta \Delta_\Phi}$.

3. HOPF-BIMODULES AND PSEUDO-MULTIPLICATIVE UNITARY

In this chapter, we recall the definition of Hopf-bimodules (3.1), the definition of a pseudo-multiplicative unitary (3.2), give the fundamental example given by groupoids (3.4), and construct the algebras and the Hopf-bimodules "generated by the left (resp. right) leg" of a pseudo-multiplicative unitary (3.3). We recall the definition of left- (resp. right-) invariant operator-valued weights on a Hopf-bimodule; if we have both operator-valued weights, we then recall Lesieur's construction of a pseudo-multiplicative unitary.

3.1. Definition. A quintuplet $(N, M, \alpha, \beta, \Gamma)$ will be called a Hopf-bimodule, following ([Val1], [EV] 6.5), if N, M are von Neumann algebras, α a faithful non-degenerate representation of N into M , β a faithful non-degenerate anti-representation of N into M , with commuting ranges, and Γ an injective involutive homomorphism from M into $M \underset{N}{\beta * \alpha} M$ such that, for all X in N :

- (i) $\Gamma(\beta(X)) = 1 \underset{N}{\beta \otimes \alpha} \beta(X)$
- (ii) $\Gamma(\alpha(X)) = \alpha(X) \underset{N}{\beta \otimes \alpha} 1$
- (iii) Γ satisfies the co-associativity relation :

$$(\Gamma \underset{N}{\beta * \alpha} id) \Gamma = (id \underset{N}{\beta * \alpha} \Gamma) \Gamma$$

This last formula makes sense, thanks to the two preceeding ones and 2.4.

If $(N, M, \alpha, \beta, \Gamma)$ is a Hopf-bimodule, it is clear that $(N^\circ, M, \beta, \alpha, \varsigma_N \circ \Gamma)$ is another Hopf-bimodule, we shall call the symmetrized of the first one. (Recall that $\varsigma_N \circ \Gamma$ is a homomorphism from M to $M \underset{N^\circ}{r * s} M$).

If N is abelian, $\alpha = \beta$, $\Gamma = \varsigma_N \circ \Gamma$, then the quadruplet $(N, M, \alpha, \alpha, \Gamma)$ is equal to its symmetrized Hopf-bimodule, and we shall say that it is a symmetric Hopf-bimodule.

Let \mathcal{G} be a groupoid, with $\mathcal{G}^{(0)}$ as its set of units, and let us denote by r and s the range and source applications from \mathcal{G} to $\mathcal{G}^{(0)}$, given by $xx^{-1} = r(x)$ and $x^{-1}x = s(x)$. As usual, we shall denote by $\mathcal{G}^{(2)}$ (or $\mathcal{G}_{s,r}^{(2)}$) the set of composable elements, i.e.

$$\mathcal{G}^{(2)} = \{(x, y) \in \mathcal{G}^2; s(x) = r(y)\}$$

In [Y] and [Val1] were associated to a measured groupoid \mathcal{G} , equipped with a Haar system $(\lambda^u)_{u \in \mathcal{G}^{(0)}}$ and a quasi-invariant measure μ on $\mathcal{G}^{(0)}$ (see [R1], [R2], [C2] II.5 and [AR] for more details, precise definitions and examples of groupoids) two Hopf-bimodules :

The first one is $(L^\infty(\mathcal{G}^{(0)}, \mu), L^\infty(\mathcal{G}, \nu), r_{\mathcal{G}}, s_{\mathcal{G}}, \Gamma_{\mathcal{G}})$, where ν is the measure constructed on \mathcal{G} using μ and the Haar system $(\lambda^u)_{u \in \mathcal{G}^{(0)}}$, where we define $r_{\mathcal{G}}$ and $s_{\mathcal{G}}$ by writing , for g in $L^\infty(\mathcal{G}^{(0)})$:

$$r_{\mathcal{G}}(g) = g \circ r$$

$$s_{\mathcal{G}}(g) = g \circ s$$

and where $\Gamma_{\mathcal{G}}(f)$, for f in $L^\infty(\mathcal{G})$, is the function defined on $\mathcal{G}^{(2)}$ by $(s, t) \mapsto f(st)$; $\Gamma_{\mathcal{G}}$ is then an involutive homomorphism from $L^\infty(\mathcal{G})$ into $L^\infty(\mathcal{G}_{s,r}^{(2)})$ (which can be identified to $L^\infty(\mathcal{G})_s * r L^\infty(\mathcal{G})$).

The second one is symmetric; it is $(L^\infty(\mathcal{G}^{(0)}), \mathcal{L}(\mathcal{G}), r_{\mathcal{G}}, r_{\mathcal{G}}, \widehat{\Gamma_{\mathcal{G}}})$, where

$\mathcal{L}(\mathcal{G})$ is the von Neumann algebra generated by the convolution algebra associated to the groupoid \mathcal{G} , and $\widehat{\Gamma}_{\mathcal{G}}$ has been defined in [Y] and [Val1].

3.2. Definition. Let N be a von Neumann algebra; let \mathfrak{H} be a Hilbert space on which N has a non-degenerate normal representation α and two non-degenerate normal anti-representations $\hat{\beta}$ and β . These 3 applications are supposed to be injective, and to commute two by two. Let ν be a normal semi-finite faithful weight on N ; we can therefore construct the Hilbert spaces $\mathfrak{H}_{\beta \otimes_{\alpha} \mathfrak{H}}$ and $\mathfrak{H}_{\alpha \otimes_{\hat{\beta}} \mathfrak{H}}$. A unitary W from $\mathfrak{H}_{\beta \otimes_{\alpha} \mathfrak{H}}$ onto $\mathfrak{H}_{\alpha \otimes_{\hat{\beta}} \mathfrak{H}}$ will be called a pseudo-multiplicative unitary over the basis N , with respect to the representation α , and the anti-representations $\hat{\beta}$ and β (we shall say it is an $(\alpha, \hat{\beta}, \beta)$ -pseudo-multiplicative unitary), if :

(i) W intertwines $\alpha, \hat{\beta}, \beta$ in the following way :

$$\begin{aligned} W(\alpha(X)_{\beta \otimes_{\alpha} 1}) &= (1_{\alpha \otimes_{\hat{\beta}} \alpha(X)})W \\ W(1_{\beta \otimes_{\alpha} \beta(X)}) &= (1_{\alpha \otimes_{\hat{\beta}} \beta(X)})W \\ W(\hat{\beta}(X)_{\beta \otimes_{\alpha} 1}) &= (\hat{\beta}(X)_{\alpha \otimes_{\hat{\beta}} 1})W \\ W(1_{\beta \otimes_{\alpha} \hat{\beta}(X)}) &= (\beta(X)_{\alpha \otimes_{\hat{\beta}} 1})W \end{aligned}$$

(ii) The operator W satisfies :

$$\begin{aligned} (1_{\mathfrak{H}_{\alpha \otimes_{\hat{\beta}} \mathfrak{H}}} W)(W_{\beta \otimes_{\alpha} 1_{\mathfrak{H}}}) &= \\ &= (W_{\alpha \otimes_{\hat{\beta}} 1_{\mathfrak{H}}}) \sigma_{\alpha, \beta}^{2,3} (W_{\beta \otimes_{\alpha} 1})(1_{\mathfrak{H}_{\beta \otimes_{\alpha} \sigma_{\nu^o}}})(1_{\mathfrak{H}_{\beta \otimes_{\alpha} W}}) \end{aligned}$$

Here, $\sigma_{\alpha, \beta}^{2,3}$ goes from $(H_{\alpha \otimes_{\hat{\beta}} H})_{\beta \otimes_{\alpha} H}$ to $(H_{\beta \otimes_{\alpha} H})_{\alpha \otimes_{\hat{\beta}} H}$, and $1_{\mathfrak{H}_{\beta \otimes_{\alpha} \sigma_{\nu^o}}}$ goes from $H_{\beta \otimes_{\alpha} (H_{\alpha \otimes_{\hat{\beta}} H})}$ to $H_{\beta \otimes_{\alpha} H_{\hat{\beta} \otimes_{\alpha} H}}$.

All the properties supposed in (i) allow us to write such a formula, which will be called the "pentagonal relation".

One should note that this definition is different from the definition introduced in [EV] (and repeated afterwards). It is in fact the same formula, the new writing

$$\sigma_{\alpha, \beta}^{2,3} (W_{\beta \otimes_{\alpha} 1})(1_{\mathfrak{H}_{\beta \otimes_{\alpha} \sigma_{\nu^o}}})$$

is here replacing the rather awkward writing

$$(\sigma_{\nu^o} \alpha \otimes_{\hat{\beta}} 1_{\mathfrak{H}})(1_{\mathfrak{H}_{\alpha \otimes_{\hat{\beta}} W}}) \sigma_{2\nu}(1_{\mathfrak{H}_{\beta \otimes_{\alpha} \sigma_{\nu^o}}})$$

but denotes the same operator, and we suggest the reader to convince himself of this easy fact.

All the properties supposed in (i) allow us to write such a formula, which will be called the "pentagonal relation".

If we start from another normal semi-finite faithful weight ν' on N , we may define, using 2.3, another unitary $W^{\nu'} = U_{\alpha, \hat{\beta}}^{\nu', \nu'^o} W U_{\beta, \alpha}^{\nu', \nu}$ from $\mathfrak{H}_{\beta \otimes_{\alpha} \mathfrak{H}}^{\nu'}$ onto $\mathfrak{H}_{\alpha \otimes_{\hat{\beta}} \mathfrak{H}}^{\nu'^o}$. The formulae which link these isomorphisms

between relative product Hilbert spaces and the relative flips allow us to check that this operator $W^{\nu'}$ is also pseudo-multiplicative; which can be resumed in saying that a pseudo-multiplicative unitary does not depend on the choice of the weight on N .

If W is an $(\alpha, \hat{\beta}, \beta)$ -pseudo-multiplicative unitary, then the unitary $\sigma_{\nu} W^* \sigma_{\nu}$ from $\mathfrak{H}_{\hat{\beta} \otimes_{\alpha} \mathfrak{H}}^{\nu}$ to $\mathfrak{H}_{\alpha \otimes_{\beta} \mathfrak{H}}^{\nu^o}$ is an $(\alpha, \beta, \hat{\beta})$ -pseudo-multiplicative unitary, called the dual of W .

3.3. Algebras and Hopf-bimodules associated to a pseudo-multiplicative

unitary. For ξ_2 in $D(\alpha \mathfrak{H}, \nu)$, η_2 in $D(\mathfrak{H}_{\hat{\beta}}, \nu^o)$, the operator $(\rho_{\eta_2}^{\alpha, \hat{\beta}})^* W \rho_{\xi_2}^{\beta, \alpha}$ will be written $(id * \omega_{\xi_2, \eta_2})(W)$; we have, therefore, for all ξ_1, η_1 in \mathfrak{H} :

$$((id * \omega_{\xi_2, \eta_2})(W) \xi_1 | \eta_1) = (W(\xi_1 \otimes_{\alpha} \xi_2) | \eta_1 \otimes_{\hat{\beta}} \eta_2)$$

and, using the intertwining property of W with $\hat{\beta}$, we easily get that $(id * \omega_{\xi_2, \eta_2})(W)$ belongs to $\hat{\beta}(N)'$.

If x belongs to N , we have $(id * \omega_{\xi_2, \eta_2})(W) \alpha(x) = (id * \omega_{\xi_2, \alpha(x) \eta_2})(W)$, and $\beta(x)(id * \omega_{\xi_2, \eta_2})(W) = (id * \omega_{\hat{\beta}(x) \xi_2, \eta_2})(W)$.

If ξ belongs to $D(\alpha \mathfrak{H}, \nu) \cap D(\mathfrak{H}_{\hat{\beta}}, \nu^o)$, we shall write $(id * \omega_{\xi})(W)$ instead of $(id * \omega_{\xi, \xi})(W)$.

We shall write $A_w(W)$ the weak closure of the linear span of these operators, which are right $\alpha(N)$ -modules and left $\beta(N)$ -modules. Applying ([E2] 3.6), we get that $A_w(W)^*$ and $A_w(W)$ are non-degenerate algebras (one should note that the notations of ([E2]) had been changed in order to fit with Lesieur's notations). We shall write $\mathcal{A}(W)$ the von Neumann algebra generated by $A_w(W)$. We then have $\mathcal{A}(W) \subset \hat{\beta}(N)'$.

For ξ_1 in $D(\mathfrak{H}_{\beta}, \nu^o)$, η_1 in $D(\alpha \mathfrak{H}, \nu)$, the operator $(\lambda_{\eta_1}^{\alpha, \hat{\beta}})^* W \lambda_{\xi_1}^{\beta, \alpha}$ will be written $(\omega_{\xi_1, \eta_1} * id)(W)$; we have, therefore, for all ξ_2, η_2 in \mathfrak{H} :

$$((\omega_{\xi_1, \eta_1} * id)(W) \xi_2 | \eta_2) = (W(\xi_1 \otimes_{\alpha} \xi_2) | \eta_1 \otimes_{\hat{\beta}} \eta_2)$$

and, using the intertwining property of W with β , we easily get that $(\omega_{\xi_1, \eta_1} * id)(W)$ belongs to $\beta(N)'$. If ξ belongs to $D(\mathfrak{H}_{\beta}, \nu^o) \cap D(\alpha \mathfrak{H}, \nu)$, we shall write $(\omega_{\xi} * id)(W)$ instead of $(\omega_{\xi, \xi} * id)(W)$.

We shall write $\widehat{A_w(W)}$ the weak closure of the linear span of these operators. It is clear that this weakly closed subspace is a non degenerate algebra; following ([EV] 6.1 and 6.5), we shall write $\widehat{\mathcal{A}(W)}$ the von Neumann algebra generated by $\widehat{A_w(W)}$. We then have $\widehat{\mathcal{A}(W)} \subset \beta(N)'$.

In ([EV] 6.3 and 6.5), using the pentagonal equation, we got that $(N, \mathcal{A}(W), \alpha, \beta, \Gamma)$, and $(N^\circ, \widehat{\mathcal{A}(W)}, \hat{\beta}, \alpha, \hat{\Gamma})$ are Hopf-bimodules, where Γ and $\hat{\Gamma}$ are defined, for any x in $\mathcal{A}(W)$ and y in $\widehat{\mathcal{A}(W)}$, by :

$$\begin{aligned}\Gamma(x) &= W^*(1_{\alpha \otimes_{\hat{\beta}} x})W \\ \hat{\Gamma}(y) &= W(y_{\beta \otimes_{\alpha} 1})W^*\end{aligned}$$

In ([EV] 6.1(iv)), we had obtained that x in $\mathcal{L}(\mathfrak{H})$ belongs to $\mathcal{A}(W)'$ if and only if x belongs to $\alpha(N)' \cap \beta(N)'$ and verifies

$$(x_{\alpha \otimes_{\hat{\beta}} 1})W = W(x_{\beta \otimes_{\alpha} 1})$$

We obtain the same way that y in $\mathcal{L}(\mathfrak{H})$ belongs to $\widehat{\mathcal{A}(W)}'$ if and only if y belongs to $\alpha(N)' \cap \hat{\beta}(N)'$ and verify $(1_{\alpha \otimes_{\hat{\beta}} y})W = W(1_{\beta \otimes_{\alpha} y})$.

Moreover, we get that $\alpha(N) \subset \mathcal{A} \cap \hat{\mathcal{A}}$, $\beta(N) \subset \mathcal{A}$, $\hat{\beta}(N) \subset \hat{\mathcal{A}}$, and, for all x in N :

$$\begin{aligned}\Gamma(\alpha(x)) &= \alpha(x)_{\beta \otimes_{\alpha} 1} \\ \Gamma(\beta(x)) &= 1_{\beta \otimes_{\alpha} \beta(x)} \\ \hat{\Gamma}(\alpha(x)) &= 1_{\alpha \otimes_{\hat{\beta}} \alpha(x)} \\ \hat{\Gamma}(\hat{\beta}(x)) &= \hat{\beta}(x)_{\alpha \otimes_{\hat{\beta}} 1}\end{aligned}$$

Following ([E2], 3.7) If η_1, ξ_2 are in $D(\alpha\mathfrak{H}, \nu)$, let us write $(id * \omega_{\xi_2, \eta_1})(\sigma_{\nu^\circ} W)$ for $(\lambda_{\eta_1}^{\alpha, \hat{\beta}})^* W \rho_{\xi_2}^{\beta, \alpha}$; we have, therefore, for all ξ_1 and η_2 in \mathfrak{H} :

$$(id * \omega_{\xi_2, \eta_1})(\sigma_{\nu^\circ} W) \xi_1 | \eta_2 = (W(\xi_1_{\beta \otimes_{\alpha} \xi_2}) | \eta_1_{\alpha \otimes_{\hat{\beta}} \eta_2})$$

Using the intertwining property of W with α , we get that it belongs to $\alpha(N)'$; we write $C_w(W)$ for the weak closure of the linear span of these operators, and we have $C_w(W) \subset \alpha(N)'$. It had been proved in ([E2], 3.10) that $C_w(W)$ is a non degenerate algebra; following ([E2] 4.1), we shall say that W is weakly regular if $C_w(W) = \alpha(N)'$. If W is weakly regular, then $A_w(W) = \mathcal{A}(W)$ and $\widehat{A_w(W)} = \widehat{\mathcal{A}(W)}$ ([E2], 3.12).

3.4. Fundamental example. Let \mathcal{G} be a measured groupoid, with $\mathcal{G}^{(0)}$ as space of units, and r and s the range and source functions from \mathcal{G} to $\mathcal{G}^{(0)}$, with a Haar system $(\lambda^u)_{u \in \mathcal{G}^{(0)}}$ and a quasi-invariant measure μ on $\mathcal{G}^{(0)}$. Let us write ν the associated measure on \mathcal{G} . Let us note :

$$\mathcal{G}_{r,r}^2 = \{(x, y) \in \mathcal{G}^2, r(x) = r(y)\}$$

Then, it has been shown [Val1] that the formula $W_{\mathcal{G}} f(x, y) = f(x, x^{-1}y)$, where x, y are in \mathcal{G} , such that $r(y) = r(x)$, and f belongs to $L^2(\mathcal{G}^{(2)})$ (with respect to an appropriate measure, constructed from λ^u and μ),

is a unitary from $L^2(\mathcal{G}^{(2)})$ to $L^2(\mathcal{G}_{r,r}^2)$ (with respect also to another appropriate measure, constructed from λ^u and μ).

Let us define $r_{\mathcal{G}}$ and $s_{\mathcal{G}}$ from $L^\infty(\mathcal{G}^{(0)})$ to $L^\infty(\mathcal{G})$ (and then considered as representations on $\mathcal{L}(L^2(\mathcal{G}))$), for any f in $L^\infty(\mathcal{G}^{(0)})$, by $r_{\mathcal{G}}(f) = f \circ r$ and $s_{\mathcal{G}}(f) = f \circ s$.

We shall identify ([Y], 3.2.2) the Hilbert space $L^2(\mathcal{G}^{(2)})$ with the relative Hilbert tensor product $L^2(\mathcal{G}, \nu) \underset{L^\infty(\mathcal{G}^{(0)}, \mu)}{s_{\mathcal{G}} \otimes_{r_{\mathcal{G}}}} L^2(\mathcal{G}, \nu)$, and the Hilbert space $L^2(\mathcal{G}_{r,r}^2)$ with $L^2(\mathcal{G}, \nu) \underset{L^\infty(\mathcal{G}^{(0)}, \mu)}{r_{\mathcal{G}} \otimes_{r_{\mathcal{G}}}} L^2(\mathcal{G}, \nu)$. Moreover, the unitary

$W_{\mathcal{G}}$ can be then interpreted [Val2] as a pseudo-multiplicative unitary over the basis $L^\infty(\mathcal{G}^{(0)})$, with respect to the representation $r_{\mathcal{G}}$, and anti-representations $s_{\mathcal{G}}$ and $r_{\mathcal{G}}$ (as here the basis is abelian, the notions of representation and anti-representations are the same, and the commutation property is fulfilled). So, we get that $W_{\mathcal{G}}$ is a $(r_{\mathcal{G}}, s_{\mathcal{G}}, r_{\mathcal{G}})$ pseudo-multiplicative unitary.

Let us take the notations of 3.3; the von Neumann algebra $\mathcal{A}(W_{\mathcal{G}})$ is equal to the von Neumann algebra $L^\infty(\mathcal{G}, \nu)$ ([Val2], 3.2.6 and 3.2.7); using ([Val2] 3.1.1), we get that the Hopf-bimodule homomorphism $\widehat{\Gamma}$ defined on $L^\infty(\mathcal{G}, \nu)$ by $W_{\mathcal{G}}$ is equal to the usual Hopf-bimodule homomorphism $\Gamma_{\mathcal{G}}$ studied in [Val1], and recalled in 3.1. Moreover, the von Neumann algebra $\widehat{\mathcal{A}(W_{\mathcal{G}})}$ is equal to the von Neumann algebra $\mathcal{L}(\mathcal{G})$ ([Val2], 3.2.6 and 3.2.7); using ([Val2] 3.1.1), we get that the Hopf-bimodule homomorphism $\widehat{\Gamma}$ defined on $\mathcal{L}(\mathcal{G})$ by $W_{\mathcal{G}}$ is the usual Hopf-bimodule homomorphism $\widehat{\Gamma}_{\mathcal{G}}$ studied in [Y] and [Val1].

Let us suppose now that the groupoid \mathcal{G} is locally compact in the sense of [R1]; it has been proved in ([E2] 4.8) that $W_{\mathcal{G}}$ is then weakly regular (in fact was proved a much stronger condition, namely the norm regularity).

3.5. Definitions ([L1], [L2]). Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf-bimodule, as defined in 3.1; a normal, semi-finite, faithful operator valued weight T from M to $\alpha(N)$ is said to be left-invariant if, for all $x \in \mathfrak{M}_T^+$, we have :

$$(id \underset{N}{\beta *_{\alpha}} T) \Gamma(x) = T(x) \underset{N}{\beta \otimes_{\alpha}} 1$$

or, equivalently (2.5), if we choose a normal, semi-finite, faithful weight ν on N , and write $\Phi = \nu \circ \alpha^{-1} \circ T$, which is a normal, semi-finite, faithful weight on M :

$$(id \underset{N}{\beta *_{\alpha}} \Phi) \Gamma(x) = T(x)$$

A normal, semi-finite, faithful operator-valued weight T' from M to $\beta(N)$ will be said to be right-invariant if it is left-invariant with respect to the symmetrized Hopf-bimodule, i.e., if, for all $x \in \mathfrak{M}_{T'}^+$, we have :

$$(T' \underset{N}{\beta *_{\alpha}} id) \Gamma(x) = 1 \underset{N}{\beta \otimes_{\alpha}} T'(x)$$

or, equivalently, if we write $\Psi = \nu \circ \beta^{-1} \circ T'$:

$$(\Psi \underset{N}{\beta * \alpha} id) \Gamma(x) = T'(x)$$

In the case of a Hopf-bimodule, with a left-invariant normal, semi-finite, faithful operator valued weight T from M to $\alpha(N)$, Lesieur had constructed an isometry U in the following way : let us choose a normal, semi-finite, faithful weight ν on N , and let us write $\Phi = \nu \circ \alpha^{-1} \circ T$, which is a normal, semi-finite, faithful weight on M ; let us write H_Φ , J_Φ , Δ_Φ for the canonical objects of the Tomita-Takesaki theory associated to the weight Φ , and let us define, for x in N , $\hat{\beta}(x) = J_\Phi \alpha(x^*) J_\Phi$. Let \mathfrak{H} be a Hilbert space on which M is acting; then ([L2], theorem 3.14), there exists an unique isometry $U_{\mathfrak{H}}$ from $\mathfrak{H}_{\alpha \otimes_{\hat{\beta}} H_\Phi}$ to $\mathfrak{H}_{\beta \otimes_\nu H_\Phi}$, such that, for any (β, ν^o) -orthogonal basis $(\xi_i)_{i \in I}$ of $(H_\Phi)_\beta$, for any a in $\mathfrak{N}_T \cap \mathfrak{N}_\Phi$ and for any v in $D((H_\Phi)_\beta, \nu^o)$, we have

$$U_{\mathfrak{H}}(v_{\alpha \otimes_{\hat{\beta}} \nu^o} \Lambda_\Phi(a)) = \sum_{i \in I} \xi_i \underset{\nu}{\beta \otimes \alpha} \Lambda_\Phi((\omega_{v, \xi_i} \underset{\nu}{\beta * \alpha} id)(\Gamma(a)))$$

Then, Lesieur proved ([L2], theorem 3.37) that, if there exists a right-invariant normal, semi-finite, faithful operator valued weight T' from M to $\beta(N)$, then the isometry U_{H_Φ} is a unitary, and that $W = U_{H_\Phi}^*$ is an $(\alpha, \hat{\beta}, \beta)$ -pseudo-multiplicative unitary from $H_{\Phi \beta \otimes_\alpha H_\Phi}$ to $H_{\Phi \alpha \otimes_{\hat{\beta}} H_\Phi}$.

Proposition *Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf-bimodule, as defined in 3.1; let us suppose that there exist a normal, semi-finite, faithful left-invariant operator valued weight T from M to $\alpha(N)$ and a right-invariant normal, semi-finite, faithful operator valued weight T' from M to $\beta(N)$; let us write $\Phi = \nu \circ \alpha^{-1} \circ T$, and let us define, for n in N :*

$$\hat{\beta}(n) = J_\Phi \alpha(n^*) J_\Phi$$

Then the $(\alpha, \hat{\beta}, \beta)$ -pseudo-multiplicative unitary from $H_{\Phi \beta \otimes_\alpha H_\Phi}$ to $H_{\Phi \alpha \otimes_{\hat{\beta}} H_\Phi}$ verifies, for any x, y_1, y_2 in $\mathfrak{N}_T \cap \mathfrak{N}_\Phi$:

$$(i * \omega_{J_\Phi \Lambda_\Phi(y_1^* y_2), \Lambda_\Phi(x)})(W) = (id \underset{N}{\beta * \alpha} \omega_{J_\Phi \Lambda_\Phi(y_2), J_\Phi \Lambda_\Phi(y_1)})(\Gamma(x^*))$$

Proof. This is just ([L2], 3.19). \square

Remark Clearly, the pseudo-multiplicative unitary W does not depend upon the choice of the right-invariant operator-valued weight T' .

4. COINVERSE AND SCALING GROUP

In this chapter, we are dealing with a Hopf-bimodule $(N, \alpha, \beta, M, \Gamma)$, equipped with a left-invariant operator-valued weight T_L , and a right-invariant operator-valued weight T_R . If ν denotes a normal semi-finite faithful weight on the basis, let Φ (resp. Ψ) be the lifted normal faithful semi-finite weight on M by T_L (resp. T_R). Then, with the additional hypothesis that the two modular automorphism groups associated to the two weight Φ and Ψ commute, we can construct a co-inverse, a scaling group and an antipod, using slight generalizations of the constructions made in ([L2],9) for "adapted measured quantum groupoids".

4.1. Definition. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N ; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M . We shall say that the weight ν is relatively invariant with respect to T_L and T_R if the two modular automorphisms groups σ^Φ and σ^Ψ commute.

4.2. Lemma. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R (4.1); we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M . Let us suppose that the two modular automorphisms groups σ^Φ and σ^Ψ commute, and let us denote δ the modulus of Ψ with respect to Φ and λ the scaling operator (2.6). We shall use the notations of 2.2.1. Then :*

(i) *let $x \in \mathcal{T}_{\Psi, T_R}$ and $n \in \mathbb{N}$ and $y = e_n x$, with the notations of 2.6; then y belongs to $\mathfrak{N}_\Psi \cap \mathfrak{N}_{T_R}$, is analytical with respect to Ψ , and the operator $\sigma_{-i/2}^\Psi(y^*)\delta^{1/2}$ is bounded, and its closure $\overline{\sigma_{-i/2}^\Psi(y^*)\delta^{1/2}}$ belongs to \mathfrak{N}_Φ ; moreover, with the identifications made in 2.6, we have :*

$$\Lambda_\Phi(\overline{\sigma_{-i/2}^\Psi(y^*)\delta^{1/2}}) = J_\Psi \Lambda_\Psi(y)$$

(ii) *let E be the linear space generated by all such elements of the form $\overline{\sigma_{-i/2}^\Psi(y^*)\delta^{1/2}}$, for all $x \in \mathcal{T}_{\Psi, T_R}$ and $n \in \mathbb{N}$; then E is a weakly dense subspace of \mathfrak{N}_Φ , and, for all $z \in E$, $\Lambda_\Phi(z) \in D((H_\Phi)_\beta, \nu^o)$;*
 (iii) *the linear set of all products $\langle \Lambda_\Phi(z), \Lambda_\Phi(z') \rangle_{\beta, \nu^o}$ (for z, z' in E) is a dense subspace of N .*

Proof. As e_n is analytical with respect to Ψ , y belongs to $\mathfrak{N}_\Psi \cap \mathfrak{N}_{T_R}$, is analytical with respect to Ψ , and $\sigma_{-i/2}^\Psi(y^*)\delta^{1/2}$ is bounded ([V], 1.2); as δ^{-1} is the modulus of Φ with respect to Ψ , we get that $\overline{\sigma_{-i/2}^\Psi(y^*)\delta^{1/2}}$ belongs to \mathfrak{N}_Φ ; we identify $\Lambda_\Phi(\overline{\sigma_{-i/2}^\Psi(y^*)\delta^{1/2}})$ with $\Lambda_\Psi(\sigma_{-i/2}^\Psi(y^*)) = J_\Psi \Lambda_\Psi(y)$, which is (i).

The subspace E contains all elements of the form $\sigma_{-i/2}^\Psi(x^*)\overline{\delta^{1/2}\sigma_{-i/2}^\Psi(e_n)}$ ($x \in \mathcal{T}_{\Psi, T_R}$), and, by density of \mathcal{T}_{Ψ, T_R} in M , we get that the closure of E contains all elements of the form $a e_n \delta^{-1/2} \delta^{1/2} \overline{\sigma_{-i/2}^\Psi(e_n)} = a e_n \sigma_{-i/2}^\Psi(e_n)$, for all $a \in M$; now, as $e_n \sigma_{-i/2}^\Psi(e_n)$ is converging to 1, we finally get that E is dense in M ; as $\Lambda_\Phi(E) \subset J_\Psi \Lambda_\Psi(\mathfrak{N}_\psi \cap \mathfrak{N}_{T_R})$, we get, by 2.2, that, for all z in E , $\Lambda_\Phi(z)$ belongs to $D((H_\Phi)_\beta, \nu^o)$; more precisely, we have :

$$R^{\beta, \nu^o}(\Lambda_\Phi(\sigma_{-i/2}^\Psi(x^*)\overline{\delta^{1/2}\sigma_{-i/2}^\Psi(e_n)})) = R^{\beta, \nu^o}(J_\psi \Lambda_\psi(e_n x)) = \Lambda_{T_R}(e_n x)$$

Therefore, the set of elements of the form $\langle \Lambda_\Phi(z), \Lambda_\Phi(z') \rangle_{\beta, \nu^o}$ contains all elements of the form $\beta^{-1} \circ T_R(x^* e_n e_n x)$, for all x in \mathcal{T}_{Ψ, T_R} and $n \in \mathbb{N}$; as $T_R(x^* e_n e_n x) = \Lambda_{T_R}(e_n x)^* \Lambda_{T_R}(e_n x) = \Lambda_{T_R}(x)^* e_n^* e_n \Lambda_{T_R}(x)$; so, its closure contains all elements of the form $\beta^{-1} \circ T_R(x^* x)$, and, therefore, it contains $\beta^{-1} \circ T_R(\mathfrak{M}_{T_R}^+)$, which finishes the proof. \square

4.3. Definition. As in ([L2], 9.2), we can define, for all $\lambda \in \mathbb{C}$, a closed operator $\Delta_\Phi^\lambda \underset{N^o}{\alpha \otimes \beta} \Delta_\Phi^\lambda$, with natural values on elementary tensor products; it is possible also to define a unitary antilinear operator $J_\Phi \underset{N^o}{\alpha \otimes \beta} J_\Phi$ from $H_\Phi \underset{N^o}{\alpha \otimes \beta} H_\Phi$ onto $H_\Phi \underset{N}{\beta \otimes \alpha} H_\Phi$ (whose inverse will be $J_\Phi \underset{N}{\beta \otimes \alpha} J_\Phi$); by composition, we define then a closed antilinear operator $S_\Phi \underset{N^o}{\alpha \otimes \beta} S_\Phi$, with natural values on elementary tensor products, whose adjoint will be $F_\Phi \underset{N}{\beta \otimes \alpha} F_\Phi$.

4.4. Proposition. For all a, c in $(\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L})^*(\mathfrak{N}_\Psi \cap \mathfrak{N}_{T_R})$, b, d in \mathcal{T}_{Ψ, T_R} and g, h in E , the following vector :

$$U_{H_\Phi}^* \Gamma(g^*) [\Lambda_\Phi(h) \underset{\nu}{\beta \otimes \alpha} (\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_\Phi((cd)^*))]$$

belongs to $D(S_\Phi \underset{\nu^*}{\alpha \otimes \beta} S_\Phi)$, and the value of $\sigma_\nu(S_\Phi \underset{\nu^*}{\alpha \otimes \beta} S_\Phi)$ on this vector is equal to :

$$U_{H_\Phi}^* \Gamma(h^*) [\Lambda_\Phi(g) \underset{\nu}{\beta \otimes \alpha} (\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_\Phi((ab)^*))]$$

Proof. The proof is identical to ([L2], 9.9), thanks to 4.2(ii). \square

4.5. Proposition. There exists a closed densely defined anti-linear operator G on H_Φ such that the linear span of :

$$(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_\Phi((cd)^*))$$

with a, c in $(\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L})^*(\mathfrak{N}_\Psi \cap \mathfrak{N}_{T_R})$, b, d in \mathcal{T}_{Ψ, T_R} , is a core of G , and we have :

$$G[(\lambda_{\Lambda_\Psi(\sigma_{\Psi_i}(b^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^\circ}{\alpha \otimes \beta} \Lambda_\Phi((cd)^*))] = (\lambda_{\Lambda_\Psi(\sigma_{\Psi_i}(d^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^\circ}{\alpha \otimes \beta} \Lambda_\Phi((ab)^*))$$

Proof. The proof is identical to ([L2], 9.10), thanks to 4.2(iii). \square

4.6. Theorem. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R ; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M . Let G be the closed densely defined antilinear operator defined in 4.5, and let $G = ID^{1/2}$ its polar decomposition. Then, the operator D is positive self-adjoint and non singular; there exists a one-parameter automorphism group τ_t on M defined, for $x \in M$, by :

$$\tau_t(x) = D^{-it} x D^{it}$$

We have, for all $n \in N$ and $t \in \mathbb{R}$:

$$\tau_t(\alpha(n)) = \alpha(\sigma_t^\nu(n))$$

$$\tau_t(\beta(n)) = \beta(\sigma_t^\nu(n))$$

which allows us to define $\tau_t \underset{N}{\beta^* \alpha} \tau_t$, $\tau_t \underset{N}{\beta^* \alpha} \sigma_t^\Phi$ and $\sigma_t^\Psi \underset{N}{\beta^* \alpha} \tau_{-t}$ on $M \underset{N}{\beta^* \alpha} M$; moreover, we have :

$$\Gamma \circ \tau_t = (\tau_t \underset{N}{\beta^* \alpha} \tau_t) \Gamma$$

$$\Gamma \circ \sigma_t^\Phi = (\tau_t \underset{N}{\beta^* \alpha} \sigma_t^\Phi) \Gamma$$

$$\Gamma \circ \sigma_t^\Psi = (\sigma_t^\Psi \underset{N}{\beta^* \alpha} \tau_{-t}) \Gamma$$

Proof. The proof is identical to [L2], 9.12 to 9.28. \square

4.7. Theorem. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R ; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M . Let G be the closed densely defined antilinear operator defined in 4.5, and let $G = ID^{1/2}$ its polar decomposition. Then, the operator I is antilinear, isometric, surjective, and we have $I = I^* = I^2$; there exists a $*$ -antiautomorphism R on M defined, for $x \in M$, by :

$$R(x) = Ix^*I$$

such that, for all $t \in \mathbb{R}$, we get $R \circ \tau_t = \tau_t \circ R$ and $R^2 = id$.
For any a, b in $\mathfrak{N}_\Psi \cap \mathfrak{N}_{T_R}$ we have :

$$R((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{N}{\beta^* \alpha} id) \Gamma(b^* b)) = (\omega_{J_\Psi \Lambda_\Psi(b)} \underset{N}{\beta^* \alpha} id) \Gamma(a^* a)$$

and for any c, d in $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$, we have :

$$R((id \underset{N}{\beta^* \alpha} \omega_{J_\Phi \Lambda_\Phi(c)}) \Gamma(d^* d)) = (id \underset{N}{\beta^* \alpha} \omega_{J_\Phi \Lambda_\Phi(d)}) \Gamma(c^* c)$$

For all $n \in N$, we have $R(\alpha(n)) = \beta(n)$, which allows us to define $R_{\underset{N}{\beta^* \alpha}} R$ from $M_{\underset{N}{\beta^* \alpha}} M$ onto $M_{\underset{N^o}{\alpha^* \beta}} M$ (whose inverse will be $R_{\underset{N^o}{\alpha^* \beta}} R$), and we have :

$$\Gamma \circ R = \varsigma_{N^o} (R_{\underset{N}{\beta^* \alpha}} R) \Gamma$$

Proof. The proof is identical to [L2], 9.38 to 9.42. \square

4.8. Theorem. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R ; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$; then :

- (i) M is the weak closure of the linear span of all elements of the form $(\omega_{\underset{N}{\beta^* \alpha}} id) \Gamma(x)$, for all $x \in M$ and $\omega \in M_*$ such that there exists $k > 0$ such that $\omega \circ \beta \leq k\nu$.
- (ii) M is the weak closure of the linear span of all elements of the form $(id \underset{N}{\beta^* \alpha} \omega) \Gamma(x)$, for all $x \in M$ and $\omega \in M_*$ such that there exists $k > 0$ such that $\omega \circ \alpha \leq k\nu$.
- (iii) M is the weak closure of the linear span of all elements of the form $(id * \omega_{v,w})(W)$, where v belongs to $D(\alpha H_\Phi, \nu)$ and w belongs to $D((H_\Phi)_{\hat{\beta}}, \nu^o)$.

Proof. The proof is identical to [L2], 9.25. \square

4.9. Definition. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R ; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M ; let τ_t the one-parameter automorphism group constructed in 4.6 and let R be the involutive $*$ -antiautomorphism constructed in 4.7. We shall call τ_t the scaling group of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$ and R the coinverse of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$. Thanks to 4.7 and 4.8, we see that, T_L and ν being given, R does not depend on the choice of the right-invariant operator-valued weight T_R , provided that there exists a right-invariant operator-valued weight T_R such that ν is relatively invariant with respect to T_L and T_R .

Similarly, from 4.6, one gets that, for all x in M , $\omega \in M_*$ such that there exists $k > 0$ with $\omega \circ \alpha \leq k\nu$, $\omega' \in M_*$ such that there exists $k > 0$ with $\omega \circ \beta \leq k\nu$, one has :

$$\tau_t((id_{\beta * \alpha} \omega) \Gamma(x)) = (id_{\beta * \alpha} \omega \circ \sigma_{-t}^\Phi) \Gamma \sigma_t^\Phi(x)$$

$$\tau_t((\omega'_{\beta * \alpha} id) \Gamma(x)) = (\omega' \circ \sigma_t^\Psi \beta * \alpha id) \Gamma \sigma_{-t}^\Psi(x)$$

So, T_L and ν being given, τ_t does not depend on the choice of the right-invariant operator-valued weight T_R , provided that there exists a right-invariant operator-valued weight T_R such that ν is relatively invariant with respect to T_L and T_R .

4.10. Theorem. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R ; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$; then, for any ξ, η in $D({}_\alpha H_\Phi, \nu) \cap D((H_\Phi)_{\hat{\beta}}, \nu^o)$, $(id * \omega_{\xi, \eta})(W)$ belongs to $D(\tau_{i/2})$, and, if we define $S = R\tau_{i/2}$, we have :*

$$S((id * \omega_{\xi, \eta})(W)) = (id * \omega_{\eta, \xi})(W)^*$$

More generally, for any x in $D(S) = D(\tau_{i/2})$, we get that $S(x)^$ belongs to $D(S)$ and $S(S(x)^*)^* = x$; S will be called the antipod of the measured quantum groupoid, and, therefore, the co-inverse and the scaling group, given by polar decomposition of the antipod, rely only upon the pseudo-multiplicative W .*

Proof. It is proved similarly to [L2] 9.35 and 9.36. \square

4.11. Proposition. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R ; let τ_t be the scaling group of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$ and R the coinverse of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$; then :*

- (i) the operator-valued weight $RT_R R$ is left-invariant, the operator valued-weight $RT_L R$ is right-invariant, and ν is relatively invariant with respect to $RT_R R$ and $RT_L R$.*
- (ii) τ_t is the scaling group of $(N, \alpha, \beta, M, \Gamma, RT_R R, RT_L R, \nu)$*

Proof. Let $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M by T_L and T_R ; the lifted weight by $RT_R R$ (resp. $RT_L R$) is then $\Psi \circ R$ (resp. $\Phi \circ R$). As $\sigma_t^{\Psi \circ R} = R \circ \sigma_{-t}^\Psi \circ R$ and $\sigma_s^{\Phi \circ R} = R \circ \sigma_{-s}^\Phi \circ R$, we get that $\sigma^{\Psi \circ R}$ and $\sigma^{\Phi \circ R}$ commute, which is (i).

From 4.6 and 4.7, we get that :

$$\begin{aligned}\Gamma \circ \sigma_t^{\Psi \circ R} &= \Gamma \circ R \circ \sigma_{-t}^{\Psi} \circ R = \varsigma_{N^\circ}(R \underset{N}{\beta} *_{\alpha} R) \Gamma \circ \sigma_{-t}^{\Psi} \circ R \\ &= \varsigma_{N^\circ}(R \circ \sigma_{-t}^{\Psi} \circ R \underset{N^\circ}{\alpha} *_{\beta} R \circ \tau_t \circ R) \varsigma_N \Gamma = (\tau_t \underset{N}{\beta} *_{\alpha} \sigma_t^{\Psi \circ R}) \Gamma\end{aligned}$$

from which we get that, for all $x \in M$ and $\omega \in M_*$ such that there exists $k > 0$ such that $\omega \circ \alpha < k\nu$, we have :

$$\tau_t((id \underset{N}{\beta} *_{\alpha} \omega) \Gamma(x)) = (id \underset{N}{\beta} *_{\alpha} \omega \circ \sigma_{-t}^{\Psi \circ R}) \Gamma(\sigma_t^{\Psi \circ R}(x))$$

from which we get, by 4.8, that τ_t is the scaling group associated to $RT_R R$, $RT_L R$ and ν . \square

5. AUTOMORPHISM GROUPS ON THE BASIS

In this section, with the same hypothesis as in chapter 4, we construct two one-parameter automorphism groups on the basis N (5.2), and we prove (5.7) that these automorphisms leave invariant the quasi-invariant weight ν . We prove also in 5.7 that the weight ν is also quasi-invariant with respect to T_L and $RT_L R$.

5.1. Lemma. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R . Let $x \in M \cap \alpha(N)'$ and $y \in M \cap \beta(N)'$. Then :*

(i) *x belongs to $\beta(N)$ if and only if we have :*

$$\Gamma(x) = 1 \underset{N}{\beta} \otimes_{\alpha} x$$

(ii) *y belongs to $\alpha(N)$ if and only if we have :*

$$\Gamma(y) = y \underset{N}{\beta} \otimes_{\alpha} 1$$

More generally, if x_1, x_2 are in $M \cap \alpha(N)'$ and such that $\Gamma(x_1) = 1 \underset{N}{\beta} \otimes_{\alpha} x_2$, then $x_1 = x_2 \in \beta(N)$.

Proof. The proof is given in [L2], 4.4. \square

5.2. Proposition. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R . Then, there exists a unique one-parameter group of automorphisms γ_t^L of N such that, for all $t \in \mathbb{R}$ and $n \in N$, we have :*

$$\begin{aligned}\sigma_t^{T_L}(\beta(n)) &= \beta(\gamma_t^L(n)) \\ \sigma_t^{RT_L R}(\alpha(n)) &= \alpha(\gamma_{-t}^L(n))\end{aligned}$$

Moreover, the automorphism groups γ^L and σ^ν commute, and there exists a positive self-adjoint non-singular operator h_L η $Z(N) \cap N^{\gamma^L}$ such that, for any $x \in N^+$ and $t \in \mathbb{R}$, we have :

$$\nu \circ \gamma_t^L(x) = \nu(h_L^t x)$$

Starting from the operator-valued weights $RT_R R$ and $RT_L R$, we obtain another one-parameter group of automorphisms γ_t^R of N , such that we have :

$$\sigma_t^{RT_R R}(\beta(n)) = \beta(\gamma_t^R(n))$$

$$\sigma_t^{T_R}(\alpha(n)) = \alpha(\gamma_{-t}^R(n))$$

and a positive self-adjoint non-singular operator h_R η $Z(N) \cap N^{\gamma^R}$ such that we have :

$$\nu \circ \gamma_t^R(x) = \nu(h_R^t x)$$

Proof. The existence of γ_t^L is given by [L2], 4.5; moreover, from the formula $\sigma_t^\Phi \circ \sigma_s^\Psi(\beta(n)) = \sigma_s^\Psi \circ \sigma_t^\Phi(\beta(n))$, we obtain :

$$\beta(\gamma_t^L \circ \sigma_{-s}^\nu(n)) = \beta(\sigma_{-s}^\nu \circ \gamma_t^L(n))$$

which gives the commutation of γ_t^L and σ_{-s}^ν . The existence of h_L is then straightforward. The construction of γ^R and h_R is just the application of the preceeding results to $RT_R R$, $RT_L R$ and ν . \square

5.3. Proposition. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R . Let T'_L (resp. T'_R) be another left (resp. right)-invariant operator-valued weight; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$, $\Phi' = \nu \circ \alpha^{-1} \circ T'_L$, $\Psi = \nu \circ \beta^{-1} \circ T_R$ and $\Psi' = \nu \circ \beta^{-1} \circ T'_R$ the lifted normal semi-finite weights on M ; then, we have :*

$$\beta(h_L^{ist}) = (D\Psi' \circ \sigma_t^\Phi : D\Psi' \circ \tau_t)_s$$

$$\alpha(h_R^{ist}) = (D\Phi' \circ \sigma_{-t}^\Psi : D\Phi' \circ \tau_t)_s$$

where τ_s is the scaling group constructed from T_L , T_R and ν as well from $RT_R R$, $RT_L R$ and ν (4.6 and 4.11).

Proof. From 4.6, we get, for all $t \in \mathbb{R}$, $\Gamma \circ \sigma_t^\Phi \tau_{-t} = (id_{\beta^* \alpha} \sigma_t^\Phi \tau_{-t}) \Gamma$, and, therefore, by the right-invariance of T'_R , we get, for all $x \in \mathfrak{M}_{T'_R}^+$, that $\tau_t \sigma_{-t}^\Phi T'_R \sigma_t^\Phi \tau_{-t}(x) = T'_R(x)$; let now $x \in \mathfrak{M}_{\Psi'}^+$; $T'_R(x)$ is an element of the positive extended part of $\beta(N)$ which can be written :

$$\int_0^\infty \lambda d e_\lambda + (1-p)\infty$$

where p is a projection in $\beta(N)$, and e_λ is a resolution of p . As x belongs to $\mathfrak{M}_{\Psi'}^+$, it is well known that $p = 1$, and $T'_R(x) = \int_0^\infty \lambda de_\lambda$. There exists also a projection q and a resolution of q such that :

$$\tau_t \sigma_{-t}^\Phi T'_R \sigma_t^\Phi \tau_{-t}(x) = \int_0^\infty \lambda df_\lambda + (1-q)\infty$$

and, for all $\mu \in \mathbb{R}^+$, we have, because $e_\mu x e_\mu$ belongs to $\mathfrak{M}_{T'_R}^+$:

$$\begin{aligned} e_\mu \left(\int_0^\infty \lambda df_\lambda \right) e_\mu + e_\mu (1-q) e_\mu \infty &= e_\mu \tau_t \sigma_{-t}^\Phi T'_R \sigma_t^\Phi \tau_{-t}(x) e_\mu \\ &= \tau_t \sigma_{-t}^\Phi T'_R \sigma_t^\Phi \tau_{-t}(e_\mu x e_\mu) \\ &= T'_R(e_\mu x e_\mu) \\ &= \int_0^\mu \lambda de_\lambda \end{aligned}$$

from which we infer that $(1-q)e_\mu = 0$, and, therefore, that $q = 1$; then, we get that $e_\mu \tau_t \sigma_{-t}^\Phi T'_R \sigma_t^\Phi \tau_{-t}(x) e_\mu$ is increasing with μ towards $T'_R(x)$. Therefore, we get that :

$$\tau_t \sigma_{-t}^\Phi T'_R \sigma_t^\Phi \tau_{-t}(x) \subset T'_R(x)$$

and, finally, the equality, for all $x \in \mathfrak{M}_{\Psi'}^+$:

$$\tau_t \sigma_{-t}^\Phi T'_R \sigma_t^\Phi \tau_{-t}(x) = T'_R(x)$$

Moreover, as we have, for all $n \in N$

$$\tau_t \sigma_{-t}^\Phi(\beta(n)) = \beta(\sigma_t^\nu \gamma_{-t}^L(n))$$

we get, using 5.2, that, for all $x \in \mathfrak{M}_{\Psi'}^+$:

$$\Psi'(\beta(h_L^{-t/2}) \sigma_t^\Phi \tau_{-t}(x) \beta(h_L^{-t/2})) = \Psi'(x)$$

and, therefore, that, for all $x \in M^+$:

$$\Psi'(\beta(h_L^{-t/2}) \sigma_t^\Phi \tau_{-t}(x) \beta(h_L^{-t/2})) \leq \Psi'(x)$$

A similar calculation (with $\tau_t \sigma_{-t}^\Phi$ instead of $\sigma_t^\Phi \tau_{-t}$) leads to :

$$\Psi'(\beta(h_L^{t/2}) \tau_t \sigma_{-t}^\Phi(x) \beta(h_L^{t/2})) \leq \Psi'(x)$$

which leads to the equality, from which we get the first result.

Applying this result to $RT_R R$, $RT_L R$ and ν , we get, using again 4.11 :

$$\begin{aligned} \beta(h_R^{ist}) &= (D\Phi' \circ R \circ \sigma_t^{\Psi \circ R} : D\Phi' \circ R \circ \tau_t)_s \\ &= (D\Phi' \circ \sigma_{-t}^\Psi \circ R : D\Phi' \circ \tau_t)_s \\ &= R[(D\Phi' \circ \sigma_{-t}^\Psi : D\Phi' \circ \tau_t)_{-s}]^* \end{aligned}$$

which leads to the result. \square

5.4. Corollary. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R . We shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M , R the coinverse and τ_t the scaling group constructed in 4.7 and 4.6; we shall denote λ the scaling operator of Ψ with respect to Φ (2.6), h_L and h_R the operators constructed in 5.2. Then, for all s, t in \mathbb{R} :*

- (i) $(D\Psi : D\Psi \circ \tau_t)_s = \lambda^{ist} \beta(h_L^{ist})$
- (ii) $(D\Phi : D\Phi \circ \tau_t)_s = \lambda^{ist} \alpha(h_R^{ist})$
- (iii) $(D\Phi : D\Phi \circ \sigma_{-t}^{\Phi \circ R})_s = \lambda^{ist} \alpha(h_R^{ist}) \alpha(h_L^{-ist})$
- (iv) $(D\Psi : D\Psi \circ \sigma_t^{\Psi \circ R})_s = \lambda^{ist} \beta(h_L^{ist}) \beta(h_R^{-ist})$.

Proof. Applying 5.3 with $T'_R = T_R$, as $(D\Psi \circ \sigma_t^\Phi : D\Psi)_s = \lambda^{-ist}$ (2.6), we obtain (i). Applying 5.3 with $T'_L = T_L$, as $(D\Phi : D\Phi \circ \sigma_{-t}^\Psi)_s = \lambda^{ist}$, we obtain (ii). Applying 5.3 with $T'_R = RT_LR$, we obtain :

$$\begin{aligned} \beta(h_L^{ist}) &= (D\Phi \circ R \circ \sigma_t^\Phi : D\Phi \circ R \circ \tau_t)_s \\ &= (D\Phi \circ \sigma_{-t}^{\Phi \circ R} \circ R : D\Phi \circ \tau_t \circ R)_s \\ &= R((D\Phi \circ \sigma_{-t}^{\Phi \circ R} : D\Phi \circ \tau_t)_{-s}^*) \end{aligned}$$

and, therefore $\alpha(h_L^{ist}) = (D\Phi \circ \sigma_{-t}^{\Phi \circ R} : D\Phi \circ \tau_t)_{-s}^*$ from which one gets :

$$\alpha(h_L^{ist}) = (D\Phi \circ \sigma_{-t}^{\Phi \circ R} : D\Phi \circ \tau_t)_s$$

Using (ii), we get :

$$(D\Phi : D\Phi \circ \sigma_{-t}^{\Phi \circ R})_s = \lambda^{ist} \alpha(h_R^{ist}) \alpha(h_L^{-ist})$$

which is (iii). And applying 5.3 with $T'_L = RT_LR$, we obtain (iv). \square

5.5. Lemma. *Let M be a von Neumann algebra, Φ a normal semi-finite faithful weight on M , θ_t a one parameter group of automorphisms of M . Let us suppose that there exists a positive non singular operator μ affiliated to M^Φ such that, for all s, t in \mathbb{R} , we have*

$$(D\Phi \circ \theta_t : D\Phi)_s = \mu^{ist}$$

We have then, for all $t \in \mathbb{R}$, $\theta_t(\mu) = \mu$. Let us write $\mu = \int_0^\infty \lambda de_\lambda$ the spectral decomposition of μ , and let us define $f_n = \int_{1/n}^n de_\lambda$. We have then, for all a in \mathfrak{N}_Φ , t in \mathbb{R} , n in \mathbb{N} :

$$\omega_{J_\Phi \Lambda_\Phi(a f_n)} \circ \theta_t = \omega_{J_\Phi \Lambda_\Phi(\theta_{-t}(a) f_n \mu^{t/2})}$$

Proof. Let us remark first that $\theta_t(\mu) = \mu$, and, therefore, $\theta_t(f_n) = f_n$. On the other hand, for any a in M , we have :

$$\theta_{-t} \sigma_s^\Phi \theta_t(x) = \sigma_s^{\Phi \circ \theta_t}(x) = \mu^{ist} \sigma_s^\Phi(x) \mu^{-ist}$$

and then :

$$\theta_{-t} \sigma_s^\Phi(x) = \mu^{ist} \sigma_s^\Phi \theta_{-t}(x) \mu^{-ist}$$

If now x is analytic with respect to Φ , we get that $\theta_{-t}(f_n x f_m)$ is analytic with respect to Φ and that :

$$f_n \theta_{-t} \sigma_{i/2}^\Phi(x) f_m = \mu^{-t/2} f_n \sigma_{i/2}^\Phi(\theta_{-t}(x)) f_m \mu^{t/2}$$

Let us take now a in \mathfrak{N}_Φ , analytic with respect to Φ ; we have, for any y in M :

$$\begin{aligned} \omega_{J_\Phi \Lambda_\Phi(f_n a f_m)} \circ \theta_t(y) &= (\theta_t(y) J_\Phi \Lambda_\Phi(f_n a f_m) | J_\Phi \Lambda_\Phi(f_n a f_m)) \\ &= (\theta_t(y) \Lambda_\Phi(f_m \sigma_{-i/2}^\Phi(a^*) f_n) | \Lambda_\Phi(f_m \sigma_{-i/2}^\Phi(a^*) f_n)) \\ &= \Phi(f_n \sigma_{i/2}^\Phi(a) f_m \theta_t(y) f_m \sigma_{-i/2}^\Phi(a^*) f_n) \end{aligned}$$

which, using the preceeding remarks, is equal to :

$$\Phi \circ \theta_t(\mu^{-t/2} f_n \sigma_{i/2}^\Phi(\theta_{-t}(a)) f_m \mu^{t/2} y \mu^{t/2} f_m \sigma_{-i/2}^\Phi(\theta_{-t}(a^*)) f_n \mu^{-t/2})$$

and, making now f_n increasing to 1, we get that $\omega_{J_\Phi \Lambda_\Phi(a f_m)} \circ \theta_t(y)$ is equal to :

$$\begin{aligned} &\Phi(\sigma_{i/2}^\Phi(\theta_{-t}(a)) f_m \mu^{t/2} y \mu^{t/2} f_m \sigma_{-i/2}^\Phi(\theta_{-t}(a^*))) \\ &= (y \Lambda_\Phi(f_m \mu^{t/2} \sigma_{-i/2}^\Phi(\theta_{-t}(a^*))) | \Lambda_\Phi(f_m \mu^{t/2} \sigma_{-i/2}^\Phi(\theta_{-t}(a^*))) \\ &= (y J_\Phi \Lambda_\Phi(\theta_{-t}(a) f_m \mu^{t/2}) | J_\Phi \Lambda_\Phi(\theta_{-t}(a) f_m \mu^{t/2})) \end{aligned}$$

from which we get the result. \square

5.6. Lemma. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R . We shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M , R the coinverse and τ_t the scaling group constructed in 4.7 and 4.6. Then, we have :*

(i) *there exists a positive non singular operator μ_1 affiliated to M^Φ and invariant under τ_t , such that $(D\Phi \circ \tau_t : D\Phi)_s = \mu_1^{ist}$; let us write $\mu_1 = \int_0^\infty \lambda d\epsilon_\lambda$ and $f_n = \int_{1/n}^n d\epsilon_\lambda$; we have then, for all a in \mathfrak{N}_Φ , t in \mathbb{R} , n in \mathbb{N} and x in M^+ :*

$$\omega_{J_\Phi \Lambda_\Phi(\tau_t(a) f_n)} = \omega_{J_\Phi \Lambda_\Phi(a f_n \mu_1^{t/2})} \circ \tau_{-t}$$

$$T \circ \tau_t(x) = \alpha \circ \sigma_t^\nu \circ \alpha^{-1}(T(\mu_1^{t/2} x \mu_1^{-t/2}))$$

(ii) *there exists a positive non singular operator μ_2 affiliated to M^Φ and invariant under $\sigma_t^{\Phi \circ R}$, such that $(D\Phi \circ \sigma_{-t}^{\Phi \circ R} : D\Phi)_s = \mu_2^{ist}$; let us write $\mu_2 = \int_0^\infty \lambda d\epsilon'_\lambda$ and $f'_n = \int_{1/n}^n d\epsilon'_\lambda$; we have then, for all b in \mathfrak{N}_Φ , t in \mathbb{R} and n in \mathbb{N} :*

$$\omega_{J_\Phi \Lambda_\Phi(b f'_n)} \circ \sigma_t^{\Phi \circ R} = \omega_{J_\Phi \Lambda_\Phi(\sigma_{-t}^{\Phi \circ R}(b) f'_n \mu_2^{-t/2})}$$

$$T(\sigma_{-t}^{\Phi \circ R}(\mu_1^{-t/2} x \mu_1^{t/2})) = \alpha \circ \gamma_t^L \circ \alpha^{-1}(T(x))$$

Moreover, we have $\mu_1^{is} = \lambda^{-is}\alpha(h_R^{-is})$, $\mu_2^{is} = \mu_1^{is}\alpha(h_L^{is})$, and μ_1^{is} , μ_2^{is} , $\alpha(h_L^{is})$ belong to $\alpha(N)' \cap M^\Phi$. The non-singular operators μ_1 , μ_2 and $\alpha(h_L)$ commute two by two.

Proof. By 5.4(ii), we get that $(D\Phi \circ \tau_t : D\Phi)_s = \lambda^{-ist}\alpha(h_R^{-ist})$, as λ is positive non singular, affiliated to the center $Z(M)$, and h_R is positive non singular affiliated to the center of N , we get there exists μ_1 positive non singular, affiliated to M^Φ such that :

$$\mu_1^{ist} = \lambda^{-ist}\alpha(h_R^{-ist}) = (D\Phi \circ \tau_t : D\Phi)_s$$

We can then apply 5.5 to τ_t and $\tau_t(a)f_n$ (which belongs to \mathfrak{N}_Φ) to get the first formula of (i). On the other hand, we get that $\alpha \circ \sigma_{-t}^\nu \circ \alpha^{-1} \circ T \circ \tau_t$ is a normal semi-finite operator-valued weight which verify, for all $x \in M^+$

$$\alpha \circ \sigma_{-t}^\nu \circ \alpha^{-1} \circ T \circ \tau_t(x) = T(\mu_1^{t/2} x \mu_1^{t/2})$$

from which we get the second formula of (i).

By 5.4(iii), we get that $(D\Phi \circ \sigma_{-t}^{\Phi \circ R} : D\Phi)_s = \lambda^{-ist}\alpha(h_R^{-ist})\alpha(h_L^{ist})$; with the same arguments, we get that there exists μ_2 positive non singular, affiliated to M^Φ such that :

$$\mu_2^{ist} = \lambda^{-ist}\alpha(h_R^{-ist})\alpha(h_L^{ist}) = (D\Phi \circ \sigma_{-t}^{\Phi \circ R} : D\Phi)_s$$

and we get the first formula of (ii) by applying again 5.5 with $\sigma_{-t}^{\Phi \circ R}$.

On the other hand, using 5.2, we get that $\alpha \circ \gamma_{-t}^L \circ \alpha^{-1} \circ T \circ \sigma_{-t}^{\Phi \circ R}$ is an operator-valued weight which verify, for all $x \in M^+$:

$$\begin{aligned} \nu \circ \alpha \circ \gamma_{-t}^L \circ \alpha^{-1} \circ T \circ \sigma_{-t}^{\Phi \circ R}(x) &= \nu(h_L^{-t/2} \alpha^{-1}(T \sigma_{-t}^{\Phi \circ R}(x)) h_L^{-t/2}) \\ &= \Phi(\alpha(h_L^{-t/2} \sigma_{-t}^{\Phi \circ R}(x) \alpha(h_L^{-t/2})) \\ &= \Phi \circ \sigma_{-t}^{\Phi \circ R}[\alpha(h_L^{-t/2}) x \alpha(h_L^{-t/2})] \\ &= \Phi(\mu_2^{t/2} \alpha(h_L^{-t/2}) x \alpha(h_L^{-t/2}) \mu_2^{t/2}) \end{aligned}$$

from which we get, because $\mu_2^{t/2} \alpha(h_L^{-t/2})$ commutes with $\alpha(N)$:

$$\alpha \circ \gamma_{-t}^L \circ \alpha^{-1} \circ T \circ \sigma_{-t}^{\Phi \circ R}(x) = T(\mu_2^{t/2} \alpha(h_L^{-t/2}) x \alpha(h_L^{-t/2}) \mu_2^{t/2})$$

or :

$$T(\sigma_{-t}^{\Phi \circ R}(x)) = \alpha \circ \gamma_t^L \circ \alpha^{-1}(T(\mu_1^{t/2} x \mu_1^{t/2}))$$

from which we finish the proof. \square

5.7. Proposition. *Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight T_L , and a right-invariant valued weight T_R ; let ν be a normal semi-finite faithful weight on N , relatively invariant with respect to T_L and T_R . We shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on M , R the coinverse and τ_t the scaling group constructed in 4.7 and 4.6; let λ be the scaling operator of Ψ with respect to Φ (2.6), γ^L and γ^R the two one-parameter automorphism groups of N introduced in 5.2*

; then, we have :

(i) for all $t \in \mathbb{R}$:

$$\Gamma \circ \tau_t = (\sigma_t^\Phi \underset{N}{\beta^* \alpha} \sigma_{-t}^{\Phi \circ R}) \Gamma = (\sigma_t^{\Psi \circ R} \underset{N}{\beta^* \alpha} \sigma_{-t}^\Psi) \Gamma$$

(ii) $h_L = h_R = 1$, and :

$$\nu \circ \gamma^L = \nu \circ \gamma^R = \nu$$

(iii) for all s, t in \mathbb{R} :

$$(D\Phi : D\Phi \circ \tau_t)_s = \lambda^{ist}$$

$$(D\Psi : D\Psi \circ \tau_t)_s = \lambda^{ist}$$

(iv) for all s, t in \mathbb{R} :

$$(D\Phi \circ \sigma_t^{\Phi \circ R} : D\Phi)_s = \lambda^{ist}$$

Therefore, the modular automorphism groups σ^Φ and $\sigma^{\Phi \circ R}$ commute, the weight ν is relatively invariant with respect to Φ and $\Phi \circ R$ and λ is the scaling operator of $\Phi \circ R$ with respect to Φ ; and we have $\tau_t(\lambda) = \lambda$, $R(\lambda) = \lambda$;

(v) there exists a non singular positive operator q affiliated to $Z(N)$ such that $\lambda = \alpha(q) = \beta(q)$.

Proof. As, for all $n \in N$, we have :

$$\sigma_{-t}^{\Phi \circ R}(\alpha(n)) = R\sigma_t^\Phi R(\alpha(n)) = \alpha(\gamma_t^L(n))$$

and, by definition, $\sigma_t^\Phi(\beta(n)) = \beta(\gamma_t^L(n))$, using a remark made in 2.4, we may consider the automorphism $\sigma_{-t}^\Phi \underset{N}{\beta^* \alpha} \sigma_t^{\Phi \circ R}$ on $M \underset{N}{\beta^* \alpha} M$; let's take a and b in $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$; let's write $h_L = \int_O^\infty \lambda de_\lambda^L$ and let us write $h_p = \int_{1/p}^p de_\lambda^L$; moreover, let's use the notations of 5.6; we have :

$$(id \underset{N}{\beta^* \alpha} \omega_{J_\Phi \Lambda_\Phi(b\alpha(h_p)f'_m)}) (\sigma_{-t}^\Phi \underset{N}{\beta^* \alpha} \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n a^* a f_n)$$

is equal to :

$$\sigma_{-t}^\Phi (id \underset{N}{\beta^* \alpha} \omega_{J_\Phi \Lambda_\Phi(b\alpha(h_p)f'_m)} \circ \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n a^* a f_n)$$

which, thanks to 5.6(ii), can be written, because $\alpha(h_p)$ belongs to $\alpha(N)' \cap M^\Phi$, and therefore $b\alpha(h_p)$ belongs to \mathfrak{N}_Φ :

$$\sigma_{-t}^\Phi (id \underset{N}{\beta^* \alpha} \omega_{J_\Phi \Lambda_\Phi(\sigma_{-t}^{\Phi \circ R}(b\alpha(h_p))f'_m \mu_2^{-t/2})}) \Gamma \circ \tau_t(f_n a^* a f_n)$$

or :

$$R\sigma_t^{\Phi \circ R} R(id \underset{N}{\beta^* \alpha} \omega_{J_\Phi \Lambda_\Phi(\sigma_{-t}^{\Phi \circ R}(b\alpha(h_p))f'_m \mu_2^{-t/2})}) \Gamma \circ \tau_t(f_n a^* a f_n)$$

By 5.6 and 2.2.2, we know that $a f_n \mu_1^{t/2}$ belongs to $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$; using now 5.6(i), we get that $\tau_t(a f_n) = \tau_t(a) f_n$ belongs to $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$.

On the other hand, by 5.6 and 2.2.2, we know that $b\alpha(h_p)f'_m$ belongs to $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$; using now 5.6(ii), we get that :

$$\sigma_{-t}^{\Phi \circ R}(b\alpha(h_p)f'_m\mu_1^{-t/2}) = \sigma_{-t}^{\Phi \circ R}(b)f'_m\mu_2^{-t/2}\alpha(h_p)\alpha(h_L^{t/2})$$

belongs to $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$, and so, using again 2.2.2,

$$\sigma_{-t}^{\Phi \circ R}(b)f'_m\mu_2^{-t/2}\alpha(h_p) = \sigma_{-t}^{\Phi \circ R}(b)f'_m\mu_2^{-t/2}\alpha(h_p)\alpha(h_L^{t/2})\alpha(h_p)\alpha(h_L^{-t/2})$$

belongs also to $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$; therefore, we can use 4.7, and we get it is equal to :

$$R\sigma_t^{\Phi \circ R}(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(\tau_t(a)f_n)}})_N(\mu_2^{-t/2}f'_m\alpha(h_p)\sigma_{-t}^{\Phi \circ R}(b^*b)\alpha(h_p)f'_m\mu_2^{-t/2})$$

which can be written, thanks to 5.6(i) :

$$R\sigma_t^{\Phi \circ R}(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(af_n\mu_1^{t/2})}})_N \circ \tau_{-t} \Gamma(\mu_2^{-t/2}f'_m\alpha(h_p)\sigma_{-t}^{\Phi \circ R}(b^*b)\alpha(h_p)f'_m\mu_2^{-t/2})$$

or, $\alpha(h_p)$, as well as $\mu_2^{-t/2}f'_m$, being invariant under $\sigma_t^{\Phi \circ R}$:

$$R(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(af_n\mu_1^{t/2})}})_N(\sigma_t^{\Phi \circ R} \beta^*_\alpha \tau_{-t}) \Gamma \circ \sigma_{-t}^{\Phi \circ R} \dots$$

$$(\mu_2^{-t/2}f'_m\alpha(h_p)b^*b\alpha(h_p)f'_m\mu_2^{-t/2})$$

and using 4.6, and again 4.7, we get it is equal to :

$$R[(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(af_n\mu_1^{t/2})}})_N \Gamma(\mu_2^{-t/2}f'_m\alpha(h_p)b^*b\alpha(h_p)f'_m\mu_2^{-t/2})]$$

$$= (id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(b\alpha(h_p)f'_m\mu_2^{-t/2})}})_N \Gamma(\mu_1^{t/2}f_n a^* a f_n \mu_1^{t/2})$$

Finally, we have proved that, for all a, b in $\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}$, m, n, p in \mathbb{N} , we have :

$$(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(b\alpha(h_p)f'_m)}})_N(\sigma_{-t}^{\Phi} \beta^*_\alpha \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n a^* a f_n) =$$

$$(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(b\alpha(h_p)f'_m\mu_2^{-t/2})}})_N \Gamma(\mu_1^{t/2}f_n a^* a f_n \mu_1^{t/2})$$

But, for all $x, y \in M$, we have :

$$\omega_{J_\Phi \Lambda_\Phi(b\alpha(h_p)f'_m)}(x) = \omega_{J_\Phi \Lambda_\Phi(b)}(\alpha(h_p)f'_m x f'_m \alpha(h_p))$$

$$\omega_{J_\Phi \Lambda_\Phi(b\alpha(h_p)f'_m\mu_2^{-t/2})}(y) = \omega_{J_\Phi \Lambda_\Phi(b)}(\alpha(h_p)f'_m\mu_2^{-t/2} x \mu_2^{-t/2} f'_m \alpha(h_p))$$

and, therefore, we get that :

$$(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(b)}})_N [(1_{\beta \otimes \alpha} \alpha(h_p)f'_m)(\sigma_{-t}^{\Phi} \beta^*_\alpha \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n a^* a f_n)(1_{\beta \otimes \alpha} f'_m \alpha(h_p))]$$

is equal to :

$$(id_{\beta^*_\alpha \omega_{J_\Phi \Lambda_\Phi(b)}})_N [(1_{\beta \otimes \alpha} \alpha(h_p)f'_m\mu_2^{-t/2}) \Gamma(\mu_1^{t/2}f_n a^* a f_n \mu_1^{t/2})(1_{\beta \otimes \alpha} \mu_2^{-t/2} f'_m \alpha(h_p))]$$

and, by density, we get that :

$$(1_{\beta \otimes \alpha} \alpha(h_p) f'_m) (\sigma_{-t}^{\Phi} \beta^* \alpha \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n a^* a f_n) (1_{\beta \otimes \alpha} f'_m \alpha(h_p))$$

is equal to :

$$(1_{\beta \otimes \alpha} \alpha(h_p) f'_m \mu_2^{-t/2}) \Gamma(\mu_1^{t/2} f_n a^* a f_n \mu_1^{t/2}) (1_{\beta \otimes \alpha} \mu_2^{-t/2} f'_m \alpha(h_p))$$

and, after making p going to ∞ , we obtain that :

$$(1_{\beta \otimes \alpha} f'_m) (\sigma_{-t}^{\Phi} \beta^* \alpha \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n a^* a f_n) (1_{\beta \otimes \alpha} f'_m)$$

is equal to (*):

$$(1_{\beta \otimes \alpha} f'_m \mu_2^{-t/2}) \Gamma(\mu_1^{t/2} f_n a^* a f_n \mu_1^{t/2}) (1_{\beta \otimes \alpha} \mu_2^{-t/2} f'_m)$$

Let's now take a file a_i in $\mathfrak{N}_{\Phi} \cap \mathfrak{N}_{T_L}$ weakly converging to 1; we get that $(1_{\beta \otimes \alpha} f'_m) (\sigma_{-t}^{\Phi} \beta^* \alpha \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n) (1_{\beta \otimes \alpha} f'_m)$ is equal to :

$$(1_{\beta \otimes \alpha} f'_m \mu_2^{-t/2}) \Gamma(\mu_1^{t/2} f_n \mu_1^{t/2}) (1_{\beta \otimes \alpha} \mu_2^{-t/2} f'_m)$$

When n goes to ∞ , then f_n is increasing to 1, the first is increasing to $1_{\beta \otimes \alpha} f'_m$, and the second is increasing to :

$$(1_{\beta \otimes \alpha} f'_m \mu_2^{-t/2}) \Gamma(\mu_1^t) (1_{\beta \otimes \alpha} \mu_2^{-t/2} f'_m)$$

which is therefore bounded.

Taking now m going to ∞ , we get that the two non-singular operators $\Gamma(\mu_1^t)$ and $1_{\beta \otimes \alpha} \mu_2^t$ are equal. Using 5.1, we get then that μ_1 is equal to μ_2 (and is affiliated to $\beta(N)$), from which we get, using 5.6, that $h_L = 1$. Applying all these calculations to $(N, \alpha, \beta, M, \Gamma, R T_R R, T_R, \nu)$, we get that $h_R = 1$, which is (ii).

Let's come back to the equality (*) above; we obtain that :

$$(1_{\beta \otimes \alpha} f'_m) (\sigma_{-t}^{\Phi} \beta^* \alpha \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(f_n a^* a f_n) (1_{\beta \otimes \alpha} f'_m)$$

is equal to :

$$(1_{\beta \otimes \alpha} f'_m) \Gamma(f_n a^* a f_n) (1_{\beta \otimes \alpha} f'_m)$$

So, when n and m go to ∞ , we obtain :

$$(\sigma_{-t}^{\Phi} \beta^* \alpha \sigma_t^{\Phi \circ R}) \Gamma \circ \tau_t(a^* a) = \Gamma(a^* a)$$

which, by density, gives the first formula of (i), the second being given then by 4.11.

From (ii) and 5.4 (i) and (ii), we get (iii).

From (ii) and 5.4(iii), we get that $(D\Phi \circ \sigma_t^{\Phi \circ R} : D\Phi)_s = \lambda^{ist}$; therefore, as λ is affiliated to $Z(M)$, we get the commutation of the modular groups σ^{Φ} and $\sigma^{\Phi \circ R}$. Using 2.6, we get that there exists λ_R positive

non singular affiliated to $Z(M)$ and δ_R positive non singular affiliated to M such that $(D\Phi \circ R : D\Phi)_t = \lambda_R^{it^2/2} \delta_R^{it}$, and the properties of R allows us to write that $R(\lambda_R) = \lambda_R$. But, on the other hand, the formula $(D\Phi \circ \sigma_t^{\Phi \circ R} : D\Phi)_s = \lambda_R^{ist}$ (2.6), gives that $\lambda_R = \lambda$ and, therefore, we get that $R(\lambda) = \lambda$. The formula $\tau_t(\lambda) = \lambda$ comes from (iii), which finishes the proof of (iv).

By (i), we have $\lambda = \mu_1 = \mu_2$, and, as we had proved that μ_1 is affiliated to $\beta(N)$, we get that λ is affiliated to $\beta(N)$; as $R(\lambda) = \lambda$ by (iv), we get (v). \square

6. MEASURED QUANTUM GROUPOIDS

In this chapter, we give a new definition (6.1) of a measured quantum groupoid, and, using [L2], we get some other results, namely on the modulus (6.3), the antipod (6.4), and the manageability of the pseudo-multiplicative unitary (6.5), all results borrowed from Lesieur.

6.1. Definition. An octuplet $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ will be called a measured quantum groupoid if :

- (i) $(N, M, \alpha, \beta, \Gamma)$ is a Hopf-bimodule
- (ii) T_L is a normal semi-finite faithful operator-valued weight from M to $\alpha(N)$, which is left-invariant, i.e. such that, for any $x \in \mathfrak{M}_{T_L}^+$:

$$(id \underset{N}{\beta * \alpha} T_L) \Gamma(x) = T_L(x) \underset{N}{\beta \otimes \alpha} 1$$

- (iii) T_R is a normal semi-finite faithful operator-valued weight from M to $\beta(N)$, which is right-invariant, i.e. such that, for any $x \in \mathfrak{M}_{T_R}^+$:

$$(T_R \underset{N}{\beta * \alpha} id) \Gamma(x) = 1 \underset{N}{\beta \otimes \alpha} T_R(x)$$

- (iv) ν is a normal semi-finite faithful weight on N , which is relatively invariant with respect to T_L and T_R , i.e. such that the modular automorphism groups σ^Φ and σ^Ψ commute, where $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$.

Let R be the co-inverse constructed in 4.7; thanks to 5.7, we get that $(N, M, \alpha, \beta, \Gamma, T_L, RT_L R, \nu)$ is a measured quantum groupoid (as well as $(N, M, \alpha, \beta, \Gamma, RT_R R, T_R, \nu)$). Moreover, R (resp. τ_t) remains the co-inverse (resp. the scaling group) of this measured quantum groupoid.

6.2. Remark. Let $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ be a measured quantum groupoid in the sense of 6.1, and let us denote R (resp. τ_t) the co-inverse (resp. the scaling group) constructed in 4.7 (resp. 4.6). Then $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ is a measured quantum groupoid in the sense of [L2], 4.1.

Conversely if $(N, M, \alpha, \beta, \Gamma, T, R, \tau, \nu)$ is a measured quantum groupoid in the sense of [L2], 4.1, then $(N, M, \alpha, \beta, \Gamma, T, RT_R, \nu)$ is a measured quantum groupoid in the sense of 6.1.

6.3. Theorem. *Let $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ be a measured quantum groupoid; let us denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$, and let R be the co-inverse and τ_t the scaling group constructed in 4.7 and 4.6. Let δ_R be the modulus of $\Phi \circ R$ with respect to Φ . Then, we have :*

- (i) $R(\delta_R) = \delta_R^{-1}$, $\tau_t(\delta_R) = \delta_R$, for all $t \in \mathbb{R}$.
- (ii) we can define a one-parameter group of unitaries $\delta_R^{it} \otimes_{\beta \otimes_{\alpha} \delta_R^{it}}^N$ which acts naturally on elementary tensor products, which verifies, for all $t \in \mathbb{R}$:

$$\Gamma(\delta_R^{it}) = \delta_R^{it} \otimes_{\beta \otimes_{\alpha} \delta_R^{it}}^N$$

Proof. Thanks to 6.2, we can rely on Lesieur's work [L2]; (i) is [L2], 5.6; (ii) is [L2], 5.20. \square

6.4. Proposition. *Let $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ be a measured quantum groupoid; let us denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$, and let R be the co-inverse and τ_t the scaling group constructed in 4.7 and 4.6. Then :*

- (i) the left ideal $\mathfrak{N}_{T_L} \cap \mathfrak{N}_{\Phi} \cap \mathfrak{N}_{R T_L R} \cap \mathfrak{N}_{\Phi \circ R}$ is dense in M , and the subspace $\Lambda_{\Phi}(\mathfrak{N}_{T_L} \cap \mathfrak{N}_{\Phi} \cap \mathfrak{N}_{R T_L R} \cap \mathfrak{N}_{\Phi \circ R})$ is dense in H_{Φ} .
- (ii) there exists a dense linear subspace $E \subset \mathfrak{N}_{\Phi}$ such that $\Lambda_{\Phi}(E)$ is dense in H_{Φ} and $J_{\Phi} \Lambda_{\Phi}(E) \subset D({}_{\alpha} H_{\Phi}, \nu) \cap D((H_{\Phi})_{\beta}, \nu^o)$.

Proof. Part (i) is given by [L2] 6.5; part (ii) by [L2] 6.7. \square

6.5. Theorem. *Let $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ be a measured quantum groupoid; let us denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$, and let R be the co-inverse and τ_t the scaling group constructed in 4.7 and 4.6. Then :*

- (i) there exists a one-parameter group of unitaries P^{it} such that, for all $t \in \mathbb{R}$ and $x \in \mathfrak{N}_{\Phi}$:

$$P^{it} \Lambda_{\Phi}(x) = \lambda^{t/2} \Lambda_{\Phi}(\tau_t(x))$$

- (ii) for any y in M , we get :

$$\tau_t(y) = P^{it} y P^{-it}$$

- (iii) we have :

$$W(P^{it} \otimes_{\beta \otimes_{\alpha} \delta_R^{it}}^N P^{it}) = (P^{it} \otimes_{\alpha \otimes_{\beta} \delta_R^{it}}^{N^o} P^{it}) W$$

- (iv) for all $v \in D(P^{-1/2})$, $w \in D(P^{1/2})$, p, q in $D({}_{\alpha} H_{\Phi}, \nu) \cap D((H_{\Phi})_{\beta}, \nu^o)$, we have :

$$(W^*(v \otimes_{\alpha \otimes_{\beta} \delta_R^{it}}^{\nu^o} q) | w \otimes_{\beta \otimes_{\alpha} \delta_R^{it}}^{\nu} p) = (W(P^{-1/2} v \otimes_{\beta \otimes_{\alpha} \delta_R^{it}}^{\nu} J_{\Phi} p) | P^{1/2} w \otimes_{\alpha \otimes_{\beta} \delta_R^{it}}^{\nu^o} J_{\Phi} q)$$

The pseudo-multiplicative unitary will be said to be "manageable", with "managing operator" P .

- (v) W is weakly regular in the sense of [E2], 4.1

Proof. The proof is given in [L2], 7.3. and 7.5. \square

6.6. Theorem. *Let $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ be a measured quantum groupoid; let us denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$, and let R be the co-inverse and τ_t the scaling group constructed in 4.7 and 4.6. Let T' be another left-invariant operator-valued weight; let us write $\Phi' = \nu \circ \alpha^{-1} \circ T'$ and let us suppose that :*

- (i) $(N, M, \alpha, \beta, \Gamma, T', RT'R, \nu)$ is a measured quantum groupoid;
- (ii) τ_t is the scaling group of this new quantum groupoid;
- (iii) for all $t \in \mathbb{R}$, the automorphism group γ'^L of N defined by $\sigma_t^{\Phi'}(\beta(n)) = \beta(\gamma'^L_t(n))$ commutes with γ^L ;

Then, there exists a strictly positive operator h affiliated to $Z(N)$ such that $(DT' : DT)_t = \beta(h^{it})$. Moreover, we have then $\gamma'^L = \gamma^L$.

Proof. This is [L2] 5.21. Then, we get :

$$\beta(\gamma'^L_t(n)) = \sigma_t^{\Phi'}(\beta(n)) = \beta(h^{-it})\beta(\gamma^L_t(n))\beta(h^{it}) = \beta(\gamma^L_t(n))$$

□

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